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APPLICATION OF GRAPH SEMIRINGS IN DECISION NETWORKS

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Abstract

Graphs satisfying some algebraic properties, specifically semiring structures are discussed with respect to their applications in some real life network problems. We illustrate some examples of the networks of graphs (where the vertices of the given network are again graphs), and use the rules of semirings to discuss their geometrical interpretations. The article leaves with an impression that such notions may be helpful in handling routing problems or optimizing the signal flows, and in joining different networks besides dealing with decision-making problems in social networks.

Keywords: Graph union and graph intersection; Beta Index; semiring; decision network

2010 Mathematics Subject Classification: 16Y60; 05C25.

1 Introduction

An ordinary network is actually an edge weighted graph $G = (V,E)$ equipped with additional information about the weights of edges and properties of the nodes. In this article, we synonymously call a graph as a network or a decision graph in accordance with the context of the usage. The graph theory has been a favorite platform for mathematicians and computer scientists for describing and analysis of the networks in a more abstract and general way. Some of the popularly known algorithms like Dijkstra algorithm, Travelling Salesman Problem (TSP) are presented in context of studying network analysis and routing problems [3]. A connection between interprocedural dataflow analysis and model checking of pushdown systems (PDSs) have been explored using semiring and its related algebraic notions [7]. In network analysis, semirings are mostly used to combine weights on the links (edges) of the network. And by doing so, we can observe different properties of network. The weights on the parallel edges are combined using the semiring addition and the weights on the sequential edges are combined using the semiring multiplication [5]. In the year 2015, Rajkumar, et al. [6] introduced a notion called *S*- valued graphs, combining the algebraic structure of semiring with that of graphs. They study graphs whose vertices and edges are assigned values from the semiring*S* (with a canonical pre-order) such that every vertex of the graph is assigned values and the weight of an edge is the minimum of the weights of the vertices incident with the edge. An approach to graph theory in an algebraic setting has been found attempted by Bustamante [1]. He used the graph operation called the linking between two graphs, which is akin to what we call *join*[1]∇, and an algebraic structure called "Link Algebra" which is analogous to the semiring$(S,\cup,\nabla)$[1]. He also used a notion called "anti-graph" such that the union of a graph and its anti-graph gives an empty graph.

Graph theory has been increasingly an active field of research in computer and mathematical sciences, and allied fields. One of the basic concepts that strikes our mind, while dealing with problems in graph theory is the connectivity of the graph. A graph *G* is said to be connected graph if there is a path between every pair of vertex. This is called the connectivity of a graph. A graph is said to be disconnected, if there exists multiple disconnected vertices and edges. Various problems in computer and mathematical sciences essentially involves the study of graph connectivity theories, namely in network applications, routing transportation networks, network tolerance etc. are few to be named. The simplest measure of the degree of connectivity of a graph is given by the Beta index It measures the level or density of connections and is defined as $\beta = \frac{|E|}{|V|}$, where $|E|$ is the total number of edges and $|V|$ is the total number of vertices in the graph or network. Trees or simple networks (without loops) have Beta value of less than one. A connected network with one cycle has a value of 1. Complex networks have a high Beta value. Interested readers may also see [3] for a brief introduction of beta index in network analysis.

2 Preliminaries

Definition 2.1. A semiring $(S,+,\cdot)$ is a non-empty set *S* equipped with two binary operations $(+)$ and $(\cdot)$ such that $(R,+)$ and $(R,\cdot)$ are semigroups and operation $(\cdot)$ distributes over $(+)$ both from the left and the right. A semiring may contain additive and multiplicative identities. When $(S,+)$ is a commutative monoid with identity 0 and $(S,\cdot)$ is a monoid with unity 1, the structure $(S,+,\cdot)$ is also called hemiring.

Definition 2.2. A semiring*S* is said to be simple if $x + 1 = 1 + x = 1$ for any $x \in S$, where 1 is the unity. *S* is said to be strict or zero sum free or anti-negative if for all $a,b \in S$, $a + b = 0$ implies that $a = 0$ and $b = 0$, where 0 is the additive identity.

[1] G. Umbrey and S. Rahman, Graph of Semirings. The article is communicated for publication.

Definition 2.3. If for all $s \in S$ and a smallest non-negative integer m, if $ms = s + s + ... + s$(m times)=0, then S is said to be of characteristic m, and if no such m exists, then S is said to be of characteristic 0.

Property 2.1. *If* $(S,+,\cdot,0,1)$ *is a commutative, simple and idempotent semirings, then S is a distributive lattice and vice versa. It is noted that every additively idempotent semiring has characteristic* 0*, this is because of the fact that* $s+s+...+s = s$ *for all* $s \in S$*. We denote the characteristic of S by char S.*

Definition 2.4. A commutative semiring with unity $(S,+,\cdot,0,1)$ is called a semifield if for all $a,b \in S$, $a + b = 0$ implies that $a = 0$ and $b = 0$, and $a \cdot b = 0$, implies that either $a = 0$ or $b = 0$.

Remark. Unlike in fields, a semifield of characteristic zero can have a finite number of elements. This can be shown with example. Let $S_k = \{0,1,2,...,k\}$ be a finite set of non-negative integers. Define two binary operations on $S_k$as for all $a,b \in S_k$, $a+b = max(a,b)$ and $a \cdot b = min(a,b)$. Then it is easy to show that $(S_k,+,\cdot,0,k)$ is a commutative and idempotent semiring with unity k. Also, if $a+b = 0$, then $a = 0$ and $b = 0$, and $a \cdot b = 0$ implies that either $a = 0$ and $b = 0$. Hence, $(S_k,+,\cdot,0,k)$ is a semifield of characteristic zero (since every(additively) idempotent semiring is of characteristic zero). Thus, this example shows that a semifield of characteristic zero may have finite number of elements, which is a major deviation of a semiring from a ring.

An undirected graph is a 2−tuppled $G = (V,E)$, where V is the set of vertices and E is the set of unordered pairs of vertices. The *union* of two graphs G_1 and G_2 is defined as the graph, $G = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$. The join ∇ of two graphs G and H is a graph formed from the copies of G and H by connecting each vertex $V(G)$ to each vertex of $V(H)$, and it is denoted by $G \nabla H = (V(G) \cup V(H), E(G) \cup E(H) \cup \{uv: u \in V(G), v \in V(H)\})$. When we ignore the self-loops, the *join* of G and H is denoted by $G \nabla H = (V(G) \cup V(H), E(G) \cup E(H) \cup \{uv: u \in V(G), v \in V(H)\} \setminus \{aa: a \in V(G) \cap V(H)\})$. Likewise, the *intersection* of two graphs G_1, and G_2 is defined as the graph, $G = (V(G_1) \cap V(G_2), E(G_1) \cap E(G_2))$. We will also use the words “conformity” and “combine” to mean graph intersection and graph union, respectively. Further notions of fundamental concepts and graph operations in graph theory can be recalled from the work of Deo [2] and Rouhonen[8]. The rank and nullity of the algebraic expressions of the graphs formed by the graphs *union* and *join* is discussed by Umbrey and Rahman [9]. Note that the graphs we consider in this article are simple and undirected.

3 On some finite semirings of graphs

We know that under the graph operations *union* and *intersection*, the set of all simple undirected graphs forms a semiring[1], which is an infinite semiring. Let Sbe the set of all

[1] G. Umbrey and S. Rahman, Graph of Semirings. The article is communicated for publication.

the subgraphs of a complete graph $K_n$with n vertices. Then it is known that formula for the number of subgraphs of $K_n$is given by [4]

$$|S| = 1 + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} \times 2^3 + \binom{n}{4} \times 2^6 + \dots + \binom{n}{n} \times 2^{\frac{n(n-1)}{2}}.$$

In view of the above formula, we can also find a large number of finite semirings of graphs as for instance, the set of all subgraphs of a complete graph K_2 on two vertices under the graph operations, namely *union* and *intersection* as addition and multiplication, respectively is a semiring of order 5. Similarly, S_{K3} and S_{K4} the set of all the subgraphs of the complete graphs K_3 and K_4, respectively endowed with *union* and *intersection* are graph semirings of order 18 and 113, respectively. Note that all these semirings are of characteristic 0.

In the family S of all possible sub graphs of G, for any $G_1, G_2 \in S$, $G_1 \subseteq G_2$ implies that there exists $G_3, G_4 \in S$ such that $G_2 = G_1 \cup G_3$ and $G_1 = G_2 \cap G_4$. Note that G_3 and G_4 may be switched here (in particular, $G_3 = G_4$). The following is a trivial example. Considering $(\emptyset,\emptyset) \subseteq G_1 \in S$ implies that $G_1 = (\emptyset,\emptyset) \cup G_2$ (in fact, $G_2 = G_1$ here) and $(\emptyset,\emptyset) = G_1 \cap G_3$ for some $G_3 \in S$. Note that the relation $\subseteq$ in S is a partial order relation, hence the set S endowed with this operation is a partial order set. The equations

$$G_2 = G_1 \cup G_3$$

and

$$G_2 = G_1 \cap G_4$$

do not have solutions if $G_2 \subset G_1$ (resp. $G_1 \subset G_2$) for all $G_3, G_4 \in S$. On the other hand, the equations $G' = (G_1 \cup G') \cap G_2$and $G' = (G_1 \cap G') \cup G_2$have solutions for all $G_1, G_2 \in S$.

Proposition 3.1. *If* $G_1 \subseteq G'$.*Then the equation* $G' = (G_1 \cap G') \cup G_2$*has solution for all* $G_2 \in S$.

Proof. Given that $G_1 \subseteq G'$. Now, $G' = (G_1 \cap G') \cup G_2 = G_1 \cup G_2$ this implies that for any $G_2 \in S$, $G_1 \subseteq G_1 \cup G_2$ i.e., $G_1 \subseteq G'$holds good for all $G_2 \in S$. □

Proposition 3.2. *If* $G' \subseteq G_1$. *Then the equation* $G' = (G_1 \cap G') \cup G_2$*has solutions for all* $G_2 \subseteq G'$*in S.*

Proof. Given that $G' \subseteq G_1$. Now, $G' = (G_1 \cap G') \cup G_2 = G' \cup G_2$, which is true for all $G' \subseteq G_1$. □

Note that in other words, if the equation $G' = (G_1 \cap G') \cup G_2$has solutions for any $G_1 \in S$, then $G' \subseteq G_1$.

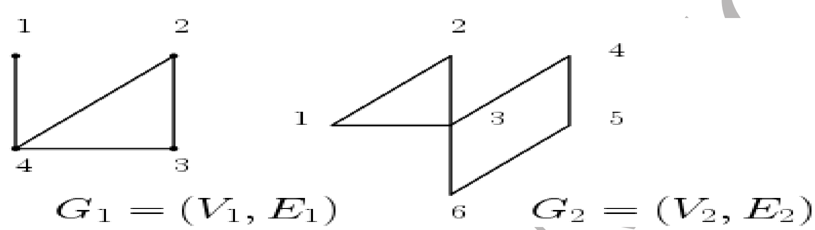
Proposition 3.3. *The equation* $G' = (G_1 \cup G') \cap G_2$ *has solutions for all* $G_2 \subseteq G_1$ *in S.*

Proof. $G' = (G_1 \cup G') \cap G_2 = (G_1 \cap G_2) \cup (G' \cap G_2) = G_2 \cup (G' \cap G_2) = G_2$. □

4 Application ofsemiring of graphs

A graph is a convenient tool for representation of real life problems, and it can capture a variety of information. We illustrate some graphs in this section and try to give some intuitive geometrical interpretations with or without using the rules of semirings.

Example 4.1. Let us consider two graphs with the number of vertices being four and five, respectively as shown below.



We consider the Beta index of the graphs G_1 and G_2 by $\beta_{G_1} = \frac{|E_1|}{|V_1|}$ and $\beta_2 = \frac{|E_2|}{|V_2|}$, respectively. In this example, $\beta_{G_1} = \frac{4}{4} = 1$ and $\beta_{G_2} = \frac{7}{6} = 1.166$. Hence G_2 is more stable than G_1.

Remark. Suppose, each vertex in a graph G represent an individual person. Two persons (or, vertices v_i and v_j) are connected by an edge (v_i,v_j) if there exists a mutual trust between v_i and v_j, that is, v_i trusts v_j and vice versa. Thus in view of this, the networks G_1 and G_2 in the example 4.1 represent the networks of mutual trusts. By comparing the beta index value of the graphs, we can conclude that G_2 has stronger mutual trusts.

Let us consider a connected network G comprising of five decision graphs (each graph being a vertex of the given network). A pair of decision graphs G_i and G_j in the given network G is connected by an edge if $G_i \cap G_j \neq (\emptyset,\emptyset)$, that is, there exists atleast a decision $G_i \cap G_j$ which is common to both G_i and G_j or the decisions of G_i and G_j has some conformity. Suppose, we want to derive the best decision out of five given decisions graphs in the network, we would collect all the edges of G. The act of collecting the edges is done with the help of the graph intersection operation, which we call semiring multiplication. Our next task in the process is to combine all the edges using the graph union operator, which we call semiring addition. Due to interplay of these two operations, endowed with the rules of semirings like associativity and distributivity, it becomes simpler to arrive at the final decision graph called solution graph, that has been contributed by all the decision graphs in the given connected network G. Finally, the stability of the solution can be checked using the Beta index of the graph. Note that the word "conformity" need not be transitive. The illustration is example 4.2.

Proposition 4.1. *In the semiring*$(S,\cup,\nabla)$ *if* $G_1 \subseteq G_2$, $G_4 \subseteq G_3$, *and* $G_1\nabla G_4 \subseteq G_2 \cup G_3$ *for some* $G_1,G_2,G_3,G_4 \in S$, *then* $(G_1\nabla G_3) \cup (G_2\nabla G_4) \subseteq G_2\nabla G_3$.

Proof. We have $G_1 \cup G_2 = G_2$, $G_3 \cup G_4 = G_3$ and $G_1\nabla G_4 \subseteq G_2 \cup G_3$. Now,

$(G_1\nabla G_3) \cup (G_2\nabla G_4)=\{G_1\nabla(G_3 \cup G_4)\} \cup \{(G_1 \cup G_2)\nabla G_4\}$
$= \{(G_1\nabla G_3) \cup (G_1\nabla G_4)\} \cup \{(G_1\nabla G_4) \cup (G_2\nabla G_4)\}$
$= (G_1\nabla G_3) \cup \{(G_1\nabla G_4) \cup (G_1\nabla G_4)\} \cup (G_2\nabla G_4)$
$= (G_1\nabla G_3) \cup (G_1\nabla G_4) \cup (G_2\nabla G_4)$
$\subseteq (G_1\nabla G_3) \cup (G_2 \cup G_3) \cup (G_2\nabla G_4)$
$\subseteq (G_1\nabla G_3) \cup (G_2\nabla G_3) \cup (G_2\nabla G_4)$(since $G_2 \cup G_3 \subseteq G_2\nabla G_3$)
$= G_2\nabla G_3$ (since $G_1\nabla G_3 \subseteq G_2\nabla G_3$ and $G_2\nabla G_4 \subseteq G_2\nabla G_3$).

□

Proposition 4.2. *In the semiring*$(S,\cup,\cap)$, *if* $G_1 \subseteq G_2$ *and* $G_4 \subseteq G_3$, *then* $(G_1 \cap G_3) \cup (G_2 \cap G_4) \subseteq G_2 \cap G_3$.

Proof. We have $G_1 \cup G_2 = G_2$ or, $G_1 \cap G_2 = G_1$, and $G_3 \cup G_4 = G_3$ or, $G_3 \cap G_4 = G_4$.
Now,

$(G_1 \cap G_3) \cup (G_2 \cap G_4) = \{(G_1 \cap G_2) \cap G_3\} \cup \{G_2 \cap (G_3 \cap G_4)\}$
$= \{(G_2 \cap G_3) \cap G_1\} \cup \{(G_2 \cap G_3) \cap G_4\}$
$= (G_2 \cap G_3) \cap (G_1 \cup G_4)$ ($\cap$ distributes over $\cup$)
$\subseteq (G_2 \cap G_3) \cap (G_2 \cup G_3)$ (since $G_1 \cup G_4 \subseteq G_2 \cup G_3$)
$= G_2 \cap G_3$.

□

Corollary 4.3. *In the semiring*$(S,\cup,\cap)$, *if* $G_1 \subseteq G_2$, $G_4 \subseteq G_3$ *and* $G_2 \cap G_3 \subseteq G_1 \cup G_4$ *then* $(G_1 \cap G_3) \cup (G_2 \cap G_4) = G_2 \cap G_3$.

Proof. We have $G_1 \cup G_2 = G_2$ or, $G_1 \cap G_2 = G_1$, and $G_3 \cup G_4 = G_3$ or, $G_3 \cap G_4 = G_4$.
Now,

$(G_1 \cap G_3) \cup (G_2 \cap G_4) = \{(G_1 \cap G_2) \cap (G_3 \cup G_4)\} \cup \{(G_1 \cup G_2) \cap (G_3 \cap G_4)\}$

$= (G_1 \cap G_2 \cap G_3) \cup (G_1 \cap G_2 \cap G_4) \cup (G_1 \cap G_3 \cap G_4) \cup$
$(G_2 \cap G_3 \cap G_4)$
$= \{(G_2 \cap G_3) \cap (G_1 \cup G_4)\} \cup \{(G_1 \cap G_4) \cap (G_2 \cup G_3)\}$
$= (G_2 \cap G_3) \cup \{(G_1 \cap G_4) \cap (G_2 \cup G_3)\}$

(Since by assumption, $G_2 \cap G_3 \subseteq G_1 \cup G_4$)

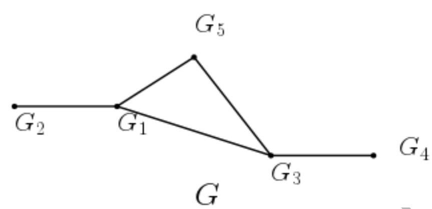
$= (G_2 \cap G_3) \cup (G_1 \cap G_4)$ (since $G_1 \cap G_4 \subseteq G_2 \cup G_3$)

$= G_2 \cap G_3$ (since $G_1 \cap G_4 \subseteq G_2 \cup G_3$).

□

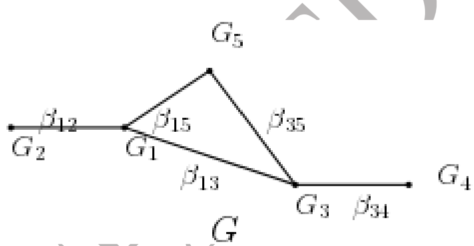
The above formulations of graph algebraic equations will help us in simplifying the graph algebraic equations in complex decision networks on one hand and enabling us to give different geometrical interpretations of the networks on the other hand. First we look at the former case through the following examples.

Example 4.2. Let us consider the following network of graphs.



Where the graphs G_1, G_2, G_3, G_4 and G_5 are from the semiring $(S, \cup, \cap)$. Find the stability of the paths connecting G_2 and G_4. Now, we propose the following simple algorithm to answer this. There are two paths connecting G_2 and G_4, namely, p_1: $G_2 - G_1 - G_3 - G_4$ and p_2: $G_2 - G_1 - G_5 - G_3 - G_4$.

Step 1: Recall that each edge is an intersection graph of the end vertices. Assign the corresponding Beta index as its weights as follows.



That is, the weight of the edge connecting the vertices (graphs) G_1 and G_2 is given by the Beta index β_{12} of intersection of the graphs G_1 and G_2, i.e., Beta index of $G_1 \cap G_2$. Likewise, the weight of the edge connecting the nodes G_1 and G_3 is given by the Beta index β_{13} of $G_1 \cap G_3$. Similarly, the weights are assigned to the remaining edges.

Step 2: Combine all the edges along the path p_1 by the means of the operation ∪, and let this combined decision graph of the path p_1 be denoted by C_{p1}. We make the following calculations to find the upper limit graph or, the super graph of C_{p1}.

C_{p1}

$$= (G_1 \cap G_2) \cup (G_1 \cap G_3) \cup (G_3 \cap G_4)$$
$$= (G_1 \cap G_2) \cup \{(G_1 \cap G_3) \cup (G_1 \cap G_3)\} \cup (G_3 \cap G_4)$$
$$= \{(G_1 \cap G_2) \cup (G_1 \cap G_3)\} \cup \{(G_1 \cap G_3) \cup (G_3 \cap G_4)\}$$
$$= \{G_1 \cap (G_2 \cup G_3)\} \cup \{G_3 \cap (G_1 \cup G_4)\}$$
$$\subseteq (G_2 \cup G_3) \cap (G_1 \cup G_4) \text{ (by proposition 4.2).}$$

Therefore, the combined decision graph of the path p_1 is a subgraph of $(G_2 \cup G_3) \cap (G_1 \cup G_4)$.

Similarly, we find the upper limit graph or, the supper graph of C_{p2} (the combined decision graph of p_2) as follows.

$$C_{p2} = (G_1 \cap G_2) \cup (G_1 \cap G_5) \cup (G_3 \cap G_5) \cup (G_3 \cap G_4)$$
$$= (G_1 \cap G_2) \cup \{(G_1 \cap G_5) \cup (G_3 \cap G_5)\} \cup \{(G_3 \cap G_5) \cup (G_3 \cap G_4)\}$$
$$= (G_1 \cap G_2) \cup [\{G_5 \cap (G_1 \cup G_3)\} \cup \{G_3 \cap (G_4 \cup G_5)\}]$$
$$\subseteq (G_1 \cap G_2) \cup (G_1 \cup G_3) \cap (G_4 \cup G_5) \text{ (by proposition 4.2).}$$

Therefore, C_{p2} is a subgraph of $(G_1 \cap G_2) \cup (G_1 \cup G_3) \cap (G_4 \cup G_5)$.

In lieu of suitable Beta index of the paths, a generalized formulation for determining the most stable path in view of the interplay of various factors in a decision making problem is discussed below.

There is no general way to compare the Beta index of a graph (V, E) and its subgraphs. Thus, to establish a numerical (or, connectivity) comparison of a graph and its subgraphs, we define a ratio, which is a function of Beta index denoted and expressed as $\boldsymbol{\beta} = 1 + \beta = 1 + \frac{|E|}{|V|}$For a discrete graph, $\boldsymbol{\beta} = 1$, while for a complete graph with n vertices, $\boldsymbol{\beta} = \frac{n+1}{2}$.Clearly, $\boldsymbol{\beta}$of a graph will be always greater than or equal to those of its subgraphs. Thus, for instance in the example 4.2, $\boldsymbol{\beta}$of p_1 will be less than or equal to that of $(G_2 \cup G_3) \cap (G_1 \cup G_4)$. Similarly, the connectivity of the rest can be compared using $\boldsymbol{\beta}$. Note that in considering the graph connectivity, the empty graph $(\emptyset,\emptyset)$ is excluded. Both β and $\boldsymbol{\beta}$are measures of connectivity of graphs. The main difference is that the least value of β and $\boldsymbol{\beta}$are 0 and 1, respectively. While, the greatest value of β and $\boldsymbol{\beta}$of a graph G with n vertices are $\frac{n-1}{2}$ and $\frac{n+1}{2}$, respectively. Unlike the beta index β, its function $\boldsymbol{\beta}$is a consistent rule to compare the connectivity of a graph with its subgraphs, which will be one of the advantages in our formulations.

Many decision problems in our real life may involve the interplay of various factors. The problems that we are considering here are presumed to be of complex in nature that may involve some vagueness also. And to deal with such varied problems, merely assigning the beta index as graph's weight may not be sufficient. Or, so to say a beta index of a graph merely tells us the intensity of the connectivity of graph, and may fail to address other parameters. For instance, in a decision making problem, if the problem is intended to determine the maximum density (degree of agreement or, conformity) of the connectivity of a complex network involving many participants without undermining the order (the number of participants) and size (conformity) of the network, then we may need more than beta index to arrive at an efficient conclusion. In such contexts, $\boldsymbol{\beta}$will be a preferred choice over β. Henceforth, we will extend the weight of each edge of the graph in the example 4.2 to $\boldsymbol{\beta}$by a transformation $\boldsymbol{\beta} = 1 + \beta$, which will be called extended weight or, simply"weight". A beautiful characteristic of $\boldsymbol{\beta}$is that it preserves the importance of order and size besides measuring its connectivity. Its value keeps on decreasing with the subsequent subgraphs, and increases or decreases proportionally with that of $\boldsymbol{\beta}$, and this property will help us to decide the most stable path (without calculation) if the resultant paths are comparable under graph order relation. That is, a super graph is a representation of more efficient decision and so forth. For instance, if the resultant paths p_1 and p_2 of the example 4.2 are non-comparable, then we will make the following calculations to find the stability of the path p_1.

Stability of the path p_1 is

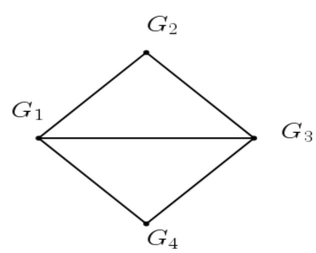
$$S_{p_1} = \frac{1}{3}\left(\sum_{i=1}^{3} \boldsymbol{\beta}_i\right)\boldsymbol{\beta}_{p_1}$$

Where$\boldsymbol{\beta}_i$is the weight of the i^{th}edge and $\boldsymbol{\beta}_{p1}$ is the weight of the upper limit graphof C_{p1}. Similarly, the stability of the path$p_2$is

$$S_{p_2} = \frac{1}{4}\left(\sum_{i=1}^{4} \boldsymbol{\beta}_i\right)\boldsymbol{\beta}_{p_2}$$

The moststable path will be the one corresponding to max(S_{p1},S_{p2}).

Example 4.3. What is the most stable path connecting the graphs G_2 and G_4 in the given network?



There are four paths connecting the graphs G_2 and G_4. Namely, p_1: $G_2 - G_1 - G_4$; p_2: $G_2 - G_1 - G_3 - G_4$; p_3: $G_2 - G_3 - G_4$; p_4: $G_2 - G_3 - G_1 - G_4$. By the similar argument as above, we can determine the most stable path.

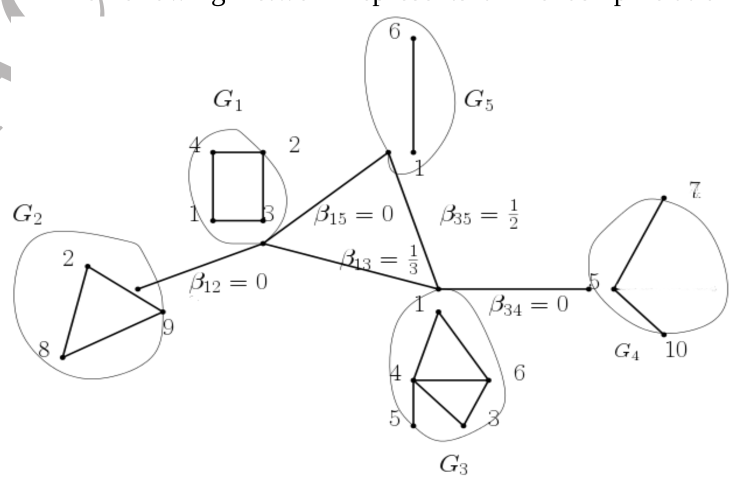
4.1 Geometrical interpretation

We are in a view that any two decision networks are comparable under subgraph relation if one represents more or less efficient decision than the other or, both represent the same level of efficiency of a decision. Under this context, we consider that a super graph represents more efficient decision than the one represented by its sub graphs. Suppose, there are two sets A and B of networks each comprising of two distinct decision networks of a same problem, namely, $G_1, G_3 \in A$ and $G_2, G_4 \in B$ such that G_2 and G_3 represent more efficient decisions than those represented by G_1 and G_4, respectively. Further, the decision represented by the conformity (intersection) of G_2 and G_3 is less efficient than the one represented by the combination (union) of G_1 and G_4. Then what would be the combined decision network of the whole sets A and B, under the condition that only conformity of G_1 and G_3 from A, and conformity of G_2 and G_4 from B are considered for final decision? The answer can be given by corollary 4.3, that is, the whole statement can be modeled mathematically as the expression on the left hand side of the equation, $(G_1 \cap G_3) \cup (G_2 \cap G_4) = G_2 \cap G_3$. The term on the right hand side of this equation is the required answer. That is, the combined decision network of the whole sets A and B is $G_2 \cap G_3$.

5 Discussions

We believe that the approaches discussed in this article will be handy to give a decision on some artificial and practical problems. For instance, let us recall the graphs in the example 4.1. Say for instance that G_1 and G_2 in the example represent two different decisions made by groups of four and six individuals, respectively, where any pair of individuals are connected by an edge if their decisions on the given problem have some conformity. In general, it is often the case in practical life that a decision will be more precise or stronger if the maximum possible number of stakeholders are consulted and most of the stakeholders have their opinions or decisions in conformity. And, so is the case we see in the example 4.1.

The following network represents a friendship relations. Each node represents an individual, and an edge between two nodes indicates that the two individuals are friends (or, they have direct friendship relation).



For instance, in G_2, all three persons have direct friendship relation. Likewise, in G_4, the individual denoted by the node 5 is a friend of both 7 and 10, but 7 and 10 have no direct friendship relations and so forth. Let us consider that the stability of a friendship network is determined by the number of friends and their intensity of connections. In the given network, we have only two choices of paths, namely, p_1: $G_2 - G_1 - G_3 - G_4$ and p_2: $G_2 - G_1 - G_5 - G_3 - G_4$. On applying the above formulae, we get the stabilities of the paths p_1 and p_2 as 0.533 and 1.687, respectively. Thus, we conclude that p_2 is more stable.

Let us also note if we take the union of all the graphs (nodes), then $G_1 \cup G_1 \cup ... \cup G_5$ would be a complex network. Now, if we are supposed to find the most stable friendship network path from 2 to 5, then it will be a tedious task for us as there would be many paths connecting these two nodes. Therefore, our approach can be seen in a way as decomposing a complex network into suitable connected components, and reducing the number of paths before applying this method.

6 Future direction and conclusion

There are many efficient algorithms available in literature that deal with shortest path algorithms and maximum capacity problems in the graph theory and computer science. This work is unique in the sense that it is a humble attempt to unite the algebraic theory and the graph theory by constructing algebraic structures (semirings) on graphs with a clear objective of its applications. Moreover, the approaches we discussed in this article are most likely different from just looking for a shortest path. Also, it is probably the first work that deals with the problems in the network of networks using the graph algebra, specifically the semirings. The limitation of this article is that we would still need to accomplish more experimental based results, and more focused scientific applications like network flow or, data routing, etc. which is left as our future research target. The ratios β and $\boldsymbol{\beta}$will be used as measures of efficiency of decisions in a varied network problems in due course of time.

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