

## FIRST KCD MATRIX AND FIRST KCD ENERGY OF A GRAPH

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### Abstract

*In this article we introduce the concept of first Karnatak College Dharwad matrix of a graph  $G$  i.e.,  $KCD_1(G)$  and related energy of a graph  $G$ . Further, first KCD polynomial of some graphs, bounds for the largest first KCD eigenvalue and first KCD energy of graphs is determined.*

**Keywords:** First KCD matrix, first KCD polynomial, first KCD eigenvalues, first KCD energy.

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### 1. Introduction

Let  $G = (V, E)$  be a simple, finite and undirected graph with  $|V(G)| = n$  as the vertex set and  $|E(G)| = m$  as the edge set.  $d_i$  is the *degree* of vertex  $v_i$  [4]. The minimum degree and maximum degree among the vertices of  $G$  is denoted as  $\delta(G)$  and  $\Delta(G)$  respectively [9].

The *adjacency matrix*  $A(G) = [a_{ij}]$  for a graph  $G$  [9] with  $n$  vertices is a  $n \times n$  matrix defined as

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j, \\ 0 & \text{otherwise.} \end{cases}$$

### First KCD matrix ..... energy of a graph

The *characteristic polynomial* of  $A(G)$  is given as  $\phi(G; \lambda) = \det(\lambda I - A(G))$ , where  $\lambda$  is a variable of degree  $n$  and  $I$  is an identity matrix. The roots of the equation  $\phi(G; \lambda) = 0$  are called the *eigenvalues* of  $G$  and are denoted as  $\lambda_1, \lambda_2, \dots, \lambda_n$ . The *spectrum*  $\text{Spec}(G)$  of a graph  $G$  is the collection of eigenvalues of  $G$  [5]. The *energy*  $E(G)$  [8] of a graph  $G$  having adjacency matrix  $A(G)$  with eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$  is defined as  $E(G) = \sum_{i=1}^n |\lambda_i|$ . The *floor function*  $[x]$  [9], for a real number  $x$  is defined as the greatest integer less than or equal to  $x$ . For undefined terminologies refer [9].

The energy concept was brought forward by Gutman in 1978 [8]. Following this, mathematical literature has received numerous contributions, like maximum degree energy of a graph [1], degree sum energy [13], degree product adjacency energy [10] and others. These energies are based on vertex degree which is interesting. The vertex degree and edge degree together makes the concept more fascinating and opens new areas of research. With this motivation we introduce *first Karnatak College Dharwad matrix*  $KCD_1(G)$  and *first Karnatak College Dharwad energy*  $E_{KCD_1}(G)$  of a graph  $G$ .

The *first KCD matrix*  $KCD_1(G)$  of a graph  $G$  is defined as

$$kcd_{1ij} = \begin{cases} (d_i + d_j) + d_e & \text{if } v_i \text{ is adjacent to } v_j, \\ 0 & \text{otherwise.} \end{cases}$$

with  $d_i$  and  $d_j$  representing degree of vertex  $v_i$  and  $v_j$  respectively,  $d_e$  is the edge degree given by  $d_e = d_i + d_j - 2$ . It is a square matrix of order  $n \times n$ .

The *first KCD polynomial* of a graph  $G$  is defined as

$$P_{KCD_1(G)}(\beta) = \det(\beta I - KCD_1(G))$$

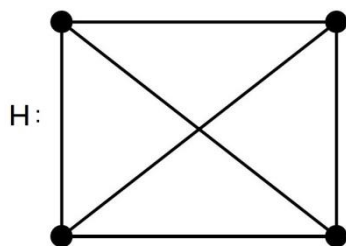
with  $\beta$  as a variable of degree  $n$  and  $I$  as an identity matrix.

$\beta_1, \beta_2, \dots, \beta_n$  represent the *first KCD eigenvalues* of  $KCD_1(G)$ . These are arranged as  $\beta_1 \geq \beta_2 \geq \dots \geq \beta_n$ , where  $\beta_1$  is the largest and  $\beta_n$  is the smallest *first KCD eigenvalue* and the collection of these *first KCD eigenvalues* is the *first KCD spectra* of  $G$ . The sum of all absolute *first KCD eigenvalues*  $\beta_1, \beta_2, \dots, \beta_n$  is called the *first KCD energy*  $E_{KCD_1}(G)$  of a graph  $G$ .

Let  $J$  be a matrix with all entries as 1 and  $I$  is an identity matrix, then for an  $r$ -regular graph  $G$  with  $n$  vertices  $KCD_1(G) = 2(2r - 1)J - 2(2r - 1)I$ .

$$\text{Thus, } P_{KCD_1(G)}(\beta) = (\beta - 2(2r - 1)(n - 1))(\beta + 2(2r - 1))^{n-1}. \quad (1.1)$$

**Example:** Let  $H = K_4$  be a complete graph. The first  $KCD$  matrix, first  $KCD$  polynomial and first  $KCD$  energy of  $H$  are as follows



**Figure 1:** Complete graph

$$KCD_1(H) = \begin{pmatrix} 0 & 10 & 10 & 10 \\ 10 & 0 & 10 & 10 \\ 10 & 10 & 0 & 10 \\ 10 & 10 & 10 & 0 \end{pmatrix}$$

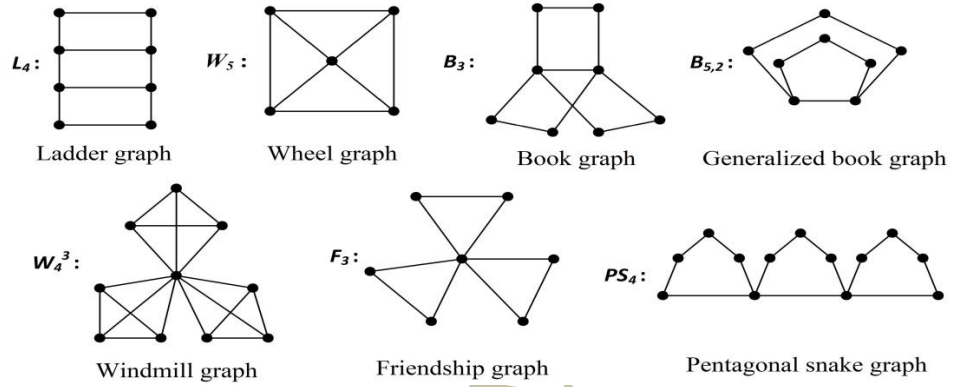
$$P_{KCD_1}(H) = \beta^4 - 600\beta^2 - 8000\beta - 30000$$

$$E_{KCD_1}(H) = 60.$$

## 2. Preliminaries

**Definition 2.1** ([7]) For  $n \geq 1$ , the **ladder graph**  $L_n$  is defined as  $L_n = P_2 \times P_n$ , with  $P_n$  as a path graph of order  $n$ . For  $n \geq 4$ , the **wheel graph**  $W_n$  of order  $n$  is the graph  $K_1 + C_{n-1}$ , with  $K_1$  as the singleton graph and  $C_{n-1}$  as the cycle graph. For  $b \geq 3$ , the **book graph**  $B_b$  is a graph defined as  $B_b = K_{1,b} \times P_2$ , with  $K_{1,b}$  as the star graph and  $P_2$  as the path graph. For  $w \geq 3$ , the **windmill graph**  $W_w^3$  is the graph formed by taking 3 copies of the complete graph  $K_w$  with a vertex in common. For  $f \geq 2$ , the graph containing  $f$  copies of cycle  $C_3$  meeting at a common vertex is the **friendship graph**  $F_f$ . For  $n \geq 2$ , the **pentagonal snake**  $PS_n$  is formed by replacing every edge of the path  $P_n$  of order  $n$  by a cycle  $C_5$ .

**Definition 2.2** ([11]) For  $p \geq 1$ , the **generalized book graph**  $B_{5,p}$  is a graph having  $p$  copies of cycle  $C_5$  with a common edge.



**Figure 2:** Examples of graphs mentioned in **Definitions 2.1** and **2.2**.

The results mentioned below are useful for computation of first *KCD* polynomial of some graphs, bounds for largest first *KCD* eigenvalue and first *KCD* energy of a graph.

**Lemma 2.3** ([14]) If  $a, b, c$  and  $d$  are real numbers, then the determinant of the form

$$\begin{vmatrix} (\beta + a)I_{n_1} - aJ_{n_1} & -cJ_{n_1 \times n_2} \\ -dJ_{n_2 \times n_1} & (\beta + b)I_{n_2} - bJ_{n_2} \end{vmatrix} \quad (2.1)$$

of order  $n_1 + n_2$  can be expressed in the simplified form as

$$(\beta + a)^{n_1-1}(\beta + b)^{n_2-1}((\beta - (n_1 - 1)a)(\beta - (n_2 - 1)b) - n_1 n_2 cd).$$

The Cauchy-Schwarz inequality [2] says, if  $(a_1, a_2, \dots, a_n)$  and  $(b_1, b_2, \dots, b_n)$  are  $n$  real vectors, then

$$\left(\sum_{i=1}^n a_i b_i\right)^2 \leq \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right) \quad (2.2)$$

**Theorem 2.4** ([12]) Let  $a_i$  and  $b_i$ ,  $1 \leq i \leq n$  are nonnegative real numbers, then

$$\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left(\sum_{i=1}^n a_i b_i\right)^2 \leq \frac{n^2}{4} (M_1 M_2 - m_1 m_2)^2 \quad (2.3)$$

where  $M_1 = \max_{1 \leq i \leq n} (a_i)$ ;  $M_2 = \max_{1 \leq i \leq n} (b_i)$ ;  $m_1 = \min_{1 \leq i \leq n} (a_i)$ ;  $m_2 = \min_{1 \leq i \leq n} (b_i)$ .

**Theorem 2.5** ([3]) Let  $a_i$  and  $b_i$ ,  $1 \leq i \leq n$  are nonnegative real numbers, then

$$\left| n \sum_{i=1}^n a_i b_i - \sum_{i=1}^n a_i \sum_{i=1}^n b_i \right| \leq \mu(n)(A - a)(B - b) \quad (2.4)$$

where  $a, b, A$  and  $B$  are real constants, such that for each  $i$ ,  $1 \leq i \leq n$ ,  $a \leq a_i \leq A$  and  $b \leq b_i \leq B$ . Further,  $\mu(n) = n \left[ \frac{n}{2} \right] \left( 1 - \frac{1}{n} \left[ \frac{n}{2} \right] \right)$ .

**Theorem 2.6** ([6]) Let  $a_i$  and  $b_i$ ,  $1 \leq i \leq n$  are nonnegative real numbers, then

$$\sum_{i=1}^n b_i^2 + c_1 c_2 \sum_{i=1}^n a_i^2 \leq (c_1 + c_2) \left( \sum_{i=1}^n a_i b_i \right) \quad (2.5)$$

where  $c_1$  and  $c_2$  are real constants, such that for each  $i$ ,  $1 \leq i \leq n$  holds,  $c_1 a_i \leq b_i \leq c_2 a_i$ .

### 3. First KCD polynomial of some graphs

**Theorem 3.1** Let  $L_n$  be a ladder graph. Then

$$P_{KCD_1(L_n)}(\beta) = (\beta + 6)^3 (\beta + 10)^{2n-5} ((\beta - 18)(\beta - 10(2n - 5)) - 256(2n - 4)).$$

**Proof.** The ladder graph  $L_n$  by definition has  $2n$  vertices. Among these  $2n$  vertices, 4 vertices have degree 2 and  $2n - 4$  vertices have degree 3.

Thus,

$$KCD_1(L_n) = \begin{bmatrix} 6(J_4 - I_4) & 8J_{4 \times (2n-4)} \\ 8J_{(2n-4) \times 4} & 10(J_{2n-4} - I_{2n-4}) \end{bmatrix}$$

and

$$\begin{aligned} P_{KCD_1(L_n)}(\beta) &= |\beta I - KCD_1(L_n)| \\ &= \begin{vmatrix} (\beta + 6)I_4 - 6J_4 & -8J_{4 \times (2n-4)} \\ -8J_{(2n-4) \times 4} & (\beta + 10)I_{2n-4} - 10J_{2n-4} \end{vmatrix}. \end{aligned}$$

Using Lemma (2.3), the desired result is obtained.  $\square$

**Illustration 3.1** Let  $L_4$  be a ladder graph. Then

$$P_{KCD_1(L_4)}(\beta) = (\beta + 6)^3 (\beta + 10)^3 ((\beta - 18)(\beta - 30) - 1024).$$

**Theorem 3.2** Let  $W_n$  be a wheel graph. Then

$$P_{KCD_1(W_n)}(\beta) = (\beta + 10)^{n-2}(\beta(\beta - 10(n - 2)) - 4(n - 1)(n + 1)^2).$$

**Proof.** The wheel graph  $W_n$  by definition has  $n$  vertices. Among these  $n$  vertices,  $n - 1$  vertices of cycle  $C_{n-1}$  have degree 3 and a central vertex has degree  $n - 1$ .

Thus,

$$KCD_1(W_n) = \begin{bmatrix} 10J_{n-1} - I_{n-1} & (2n + 2)J_{(n-1) \times 1} \\ (2n + 2)J_{1 \times (n-1)} & J_1 - I_1 \end{bmatrix}$$

and

$$\begin{aligned} P_{KCD_1(W_n)}(\beta) &= |\beta I - KCD_1(W_n)| \\ &= \begin{vmatrix} (\beta + 10)J_{n-1} - 10J_{n-1} & -(2n + 2)J_{(n-1) \times 1} \\ -(2n + 2)J_{1 \times (n-1)} & (\beta + 1)I_1 - J_1 \end{vmatrix}. \end{aligned}$$

Using Lemma (2.3), the desired result is obtained. □

**Illustration 3.2** Let  $W_5$  be a wheel graph. Then

$$P_{KCD_1(W_5)}(\beta) = (\beta + 10)^3(\beta(\beta - 30) - 576).$$

**Theorem 3.3** Let  $B_b$  be a book graph. Then

$$\begin{aligned} P_{KCD_1(B_b)}(\beta) &= (\beta + 6)^{2b-1}(\beta + 4b + 2)((\beta - 6(2b - 1))(\beta - (4b + 2)) \\ &\quad - 16b(b + 2)^2). \end{aligned}$$

**Proof.** The book graph  $B_b$  by definition has  $2b + 2$  vertices. Among these  $2b + 2$  vertices,  $2b$  vertices have degree 2 and 2 vertices have degree  $b + 1$ .

Thus,

$$KCD_1(B_b) = \begin{bmatrix} 6(J_{2b} - I_{2b}) & (2b + 4)J_{2b \times 2} \\ (2b + 4)J_{2 \times 2b} & (4b + 2)(J_2 - I_2) \end{bmatrix}$$

and

$$\begin{aligned} P_{KCD_1(B_b)}(\beta) &= |\beta I - KCD_1(B_b)| \\ &= \begin{vmatrix} (\beta + 6)I_{2b} - 6J_{2b} & -(2b + 4)J_{2b \times 2} \\ -(2b + 4)J_{2 \times 2b} & (\beta + (4b + 2))I_2 - (4b + 2)J_2 \end{vmatrix}. \end{aligned}$$

Using Lemma (2.3), the desired result is obtained. □

**Illustration 3.3** Let  $B_3$  be a book graph. Then

$$P_{KCD_1(B_3)}(\beta) = (\beta + 6)^5(\beta + 14)((\beta - 30)(\beta - 14) - 1200).$$

**Theorem 3.4** Let  $W_w^3$  be a windmill graph. Then

$$P_{KCD_1(W_w^3)}(\beta) = (\beta + 4w - 6)^{3w-4}(\beta(\beta - (3w - 4)(4w - 6)) - 12(w - 1)(4w - 5)^2).$$

**Proof.** The windmill graph  $W_w^3$  by definition has  $3w - 2$  vertices. Among these  $3w - 2$  vertices,  $3w - 3$  vertices have degree  $w - 1$  and one vertex has degree  $3(w - 1)$ .

Thus,

$$KCD_1(W_w^3) = \begin{bmatrix} (4w - 6)(J_{3w-3} - I_{3w-3}) & (8w - 10)J_{(3w-3) \times 1} \\ (8w - 10)J_{1 \times (3w-3)} & J_1 - I_1 \end{bmatrix}$$

and

$$\begin{aligned} P_{KCD_1(W_w^3)}(\beta) &= |\beta I - KCD_1(W_w^3)| \\ &= \begin{vmatrix} (\beta + (4w - 6))I_{3w-3} - (4w - 6)J_{3w-3} & -(8w - 10)J_{(3w-3) \times 1} \\ -(8w - 10)J_{1 \times (3w-3)} & (\beta + 1)I_1 - J_1 \end{vmatrix}. \end{aligned}$$

Using Lemma (2.3), the desired result is obtained. □

**Illustration 3.4** Let  $W_4^3$  be a windmill graph. Then

$$P_{KCD_1(W_4^3)}(\beta) = (\beta + 10)^8(\beta(\beta - 80) - 4356).$$

**Theorem 3.5** Let  $F_f$  be a friendship graph. Then

$$P_{KCD_1(F_f)}(\beta) = (\beta + 6)^{2f-1}(\beta(\beta - 6(2f - 1)) - 8f(2f + 1)^2).$$

**Proof.** The friendship graph  $F_f$  by definition has  $2f + 1$  vertices. Among these  $2f + 1$  vertices,  $2f$  vertices have degree 2 and one vertex has degree  $2f$ .

Thus,

$$KCD_1(F_f) = \begin{bmatrix} 6(J_{2f} - I_{2f}) & (4f + 2)J_{2f \times 1} \\ (4f + 2)J_{1 \times 2f} & J_1 - I_1 \end{bmatrix}$$

and

$$\begin{aligned} P_{KCD_1(F_f)}(\beta) &= |\beta I - KCD_1(F_f)| \\ &= \begin{vmatrix} (\beta + 6)I_{2f} - 6J_{2f} & -(4f + 2)J_{2f \times 1} \\ -(4f + 2)J_{1 \times 2f} & (\beta + 1)I_1 - J_1 \end{vmatrix}. \end{aligned}$$

Using Lemma (2.3), the desired result is obtained.

□

**Illustration 3.5** Let  $F_3$  be a friendship graph. Then

$$P_{KCD_1(F_3)}(\beta) = (\beta + 6)^5(\beta(\beta - 30) - 1176).$$

**Theorem 3.6** Let  $PS_n$  be a pentagonal snake graph. Then

$$\begin{aligned} P_{KCD_1(PS_n)}(\beta) &= (\beta + 6)^{3n-2}(\beta + 14)^{n-3}((\beta - 6(3n - 2))(\beta - 14(n - 3)) \\ &\quad - 100(3n - 1)(n - 2)). \end{aligned}$$



**Proof.** The pentagonal snake  $PS_n$  by definition has  $4n - 3$  vertices. Among these  $4n - 3$  vertices,  $3n - 1$  vertices have degree 2 and  $n - 2$  vertices have degree 4.

Thus,

$$KCD_1(PS_n) = \begin{bmatrix} 6J_{3n-1} - I_{3n-1} & 10J_{(3n-1) \times (n-2)} \\ 10J_{(n-2) \times (3n-1)} & 14(J_{n-2} - I_{n-2}) \end{bmatrix}$$

and

$$\begin{aligned} P_{KCD_1(PS_n)}(\beta) &= |\beta I - KCD_1(PS_n)| \\ &= \begin{vmatrix} (\beta + 6)I_{3n-1} - 6J_{3n-1} & -10J_{(3n-1) \times (n-2)} \\ -10J_{(n-2) \times (3n-1)} & (\beta + 14)I_{n-2} - 14J_{n-2} \end{vmatrix}. \end{aligned}$$

Using Lemma (2.3), the desired result is obtained. □

**Illustration 3.6** Let  $PS_4$  be a pentagonal snake graph. Then

$$P_{KCD_1(PS_4)}(\beta) = (\beta + 6)^{10}(\beta + 14)((\beta - 60)(\beta - 14) - 2200).$$

**Theorem 3.7** Let  $B_{5,p}$  be a generalized book graph. Then

$$\begin{aligned} P_{KCD_1(B_{5,p})}(\beta) &= (\beta + 6)^{3p-1}(\beta + 4p + 2)((\beta - 6(3p - 1))(\beta - (4p + 2)) \\ &\quad - 24p(p + 2)^2). \end{aligned}$$

**Proof.** The generalized book graph  $B_{5,p}$  by definition has  $3p + 2$  vertices. Among these  $3p + 2$  vertices,  $3p$  vertices have degree 2 and 2 vertices have degree  $p + 1$ .

Thus,

$$KCD_1(B_{5,p}) = \begin{bmatrix} 6(J_{3p} - I_{3p}) & (2p + 4)J_{3p \times 2} \\ (2p + 4)J_{2 \times 3p} & (4p + 2)(J_2 - I_2) \end{bmatrix}$$

and

$$P_{KCD_1(B_{5,p})}(\beta) = |\beta I - KCD_1(B_{5,p})|$$

$$= \begin{vmatrix} (\beta + 6)I_{3p} - 6J_{3p} & -(2p + 4)J_{3p \times 2} \\ -(2p + 4)J_{2 \times 3p} & (\beta + (4p + 2))I_2 - (4p + 2)J_2 \end{vmatrix}.$$

Using Lemma (2.3), the desired result is obtained.  $\square$

**Illustration 3.7** Let  $B_{5,2}$  be a generalized book graph. Then

$$P_{KCD_1(B_{5,2})}(\beta) = (\beta + 6)^5(\beta + 10)((\beta - 30)(\beta - 10) - 768).$$

#### 4. Bounds for the largest first $KCD$ eigenvalue and first $KCD$ energy

**Theorem 4.1** The eigenvalues of  $KCD_1(G)$  satisfies the relations

1.  $\sum_{i=1}^n \beta_i = 0$
2.  $\sum_{i=1}^n \beta_i^2 = 2Q$ , where  $Q = \sum_{i < j} 4(d_i + d_j - 1)^2$ .

**Proof.** By the definition of  $KCD_1(G)$ ,

$$\sum_{i=1}^n \beta_i = 0. \tag{4.1}$$

Further,

$$\begin{aligned} \sum_{i=1}^n \beta_i^2 &= \text{trace}((KCD_1(G))^2) \\ &= \sum_{i=1}^n \sum_{j=1}^n d_{ij}d_{ji} \\ &= \sum_{i=1}^n \sum_{j=1}^n d_{ij}^2 \\ &= 2 \sum_{i < j} ((d_i + d_j) + d_e)^2, \text{ where } d_e = d_i + d_j - 2 \\ &= 2 \sum_{i < j} 4(d_i + d_j - 1)^2 \\ &= 2Q, \text{ where } Q = \sum_{i < j} 4(d_i + d_j - 1)^2. \end{aligned} \tag{4.2}$$

$\square$

**Illustration 4.1** For the graph  $H$  in the Figure 1,  $\sum_{i=1}^4 \beta_i = 0$  and  $\sum_{i=1}^4 \beta_i^2 = 2Q = 1200$ .

**Theorem 4.2** *If  $G$  is a graph with  $n$  vertices, then*

$$\beta_1 \leq \sqrt{\frac{2Q(n-1)}{n}}. \quad (4.3)$$

**Proof.** Let  $a_i = 1$  and  $b_i = \beta_i$  for  $i = 1, 2, \dots, n$  in inequality (2.2)

then,

$$(\sum_{i=1}^n \beta_i)^2 \leq (n-1)(\sum_{i=1}^n \beta_i^2) \quad (4.4)$$

From Eqs. (4.1) and (4.2), we get

$$\sum_{i=2}^n \beta_i = -\beta_1 \quad \text{and} \quad \sum_{i=2}^n \beta_i^2 = 2Q - \beta_1^2.$$

Thus inequality (4.4) implies,

$$(-\beta_1^2) \leq (n-1)(2Q - \beta_1^2).$$

Hence,

$$\beta_1 \leq \sqrt{\frac{2Q(n-1)}{n}}.$$

Equality for  $\beta_1$  holds if graph  $G$  is regular. □

**Illustration 4.2** Consider the graph  $W_5$  in the Figure 2. It has  $\beta_1 = 36$  and  $2Q = 1952$ , therefore satisfies inequality in Theorem 4.2. Further, for the regular graph  $H$  in the Figure 1,  $\beta_1 = 30$  and  $2Q = 1200$ , thus satisfying the equality in the Theorem 4.2.

**Theorem 4.3** *If  $G$  is a graph with  $n$  vertices, then*

$$\sqrt{2Q} \leq E_{KCD_1}(G) \leq \sqrt{2nQ}.$$

**Proof.** For  $a_i = 1$  and  $b_i = \beta_i$  in inequality (2.2)

we obtain,

$$(\sum_{i=1}^n |\beta_i|)^2 \leq n(\sum_{i=1}^n |\beta_i|^2)$$

**First  $KCD$  matrix ..... energy of a graph**

Using definition of first  $KCD$  energy of a graph  $G$  and Eq. (4.2), we get

$$(E_{KCD_1}(G))^2 \leq 2nQ.$$

Thus,

$$E_{KCD_1}(G) \leq \sqrt{2nQ}. \quad (4.5)$$

Since,

$$(E_{KCD_1}(G))^2 = (\sum_{i=1}^n |\beta_i|)^2 \geq \sum_{i=1}^n |\beta_i|^2 = 2Q.$$

Thus,

$$E_{KCD_1}(G) \geq \sqrt{2Q}. \quad (4.6)$$

From Eqs. (4.5) and (4.6), required result is generated.

□

**Illustration 4.3** Consider the graph  $W_5$  in the Figure 2. It has  $E_{KCD_1}(W_5) = 72$ . Further,  $2Q = 1952$  and  $2nQ = 9760$ , therefore satisfying the Theorem 4.3.

**Theorem 4.4** If  $G$  is a graph with  $n$  vertices, then

$$E_{KCD_1}(G) \geq \sqrt{2nQ - \frac{n^2}{4} (|\beta_1| - |\beta_n|)^2}.$$

where  $|\beta_1|$  is maximum and  $|\beta_n|$  is minimum of the absolute value of  $\beta_i$ 's.

**Proof.** For  $a_i = 1$  and  $b_i = \beta_i$  in inequality (2.3)

we obtain,

$$\sum_{i=1}^n 1^2 \sum_{i=1}^n |\beta_i|^2 - (\sum_{i=1}^n |\beta_i|)^2 \leq \frac{n^2}{4} (|\beta_1| - |\beta_n|)^2$$

$$2nQ - (E_{KCD_1}(G))^2 \leq \frac{n^2}{4} (|\beta_1| - |\beta_n|)^2$$

$$E_{KCD_1}(G) \geq \sqrt{2nQ - \frac{n^2}{4} (|\beta_1| - |\beta_n|)^2}.$$

□

**Illustration 4.4** Consider the graph  $H$  in the Figure 1. It has  $|\beta_1| = 30$ ,  $|\beta_4| = 10$ ,  $2nQ = 4800$ ,  $n = 4$ . Further  $E_{KCD_1}(H) = 60$ . Therefore it satisfies the Theorem 4.4.

**Theorem 4.5** If  $G$  is a  $(n, m)$  graph, then

$$E_{KCD_1}(G) \geq \sqrt{2nQ - \mu(n)(|\beta_1| - |\beta_n|)^2}$$

where  $\mu(n) = n \binom{n}{2} \left(1 - \frac{1}{n} \binom{n}{2}\right)$ .

**Proof.** Let  $a_i = |\beta_i| = b_i$ ,  $A = |\beta_1| = B$  and  $a = |\beta_n| = b$  in inequality (2.4)

then,

$$|n \sum_{i=1}^n |\beta_i|^2 - (\sum_{i=1}^n |\beta_i|)^2| \leq \mu(n)(|\beta_1| - |\beta_n|)^2 \quad (4.7)$$

Since,

$$E_{KCD_1}(G) = \sum_{i=1}^n |\beta_i| \quad \text{and} \quad \sum_{i=1}^n |\beta_i|^2 = 2Q.$$

Inequality (4.7) gives

$$2nQ - (E_{KCD_1}(G))^2 \leq \mu(n)(|\beta_1| - |\beta_n|)^2. \quad (4.8)$$

Simplification of inequality (4.8) generates desired result.

□

**Illustration 4.5** For the graph  $H$  in the Figure 1,  $|\beta_1| = 30$ ,  $|\beta_4| = 10$ ,  $2nQ = 4800$ ,  $n = 4$ ,  $\mu(n) = 4$ . Further  $E_{KCD_1}(H) = 60$ . Thus it satisfies the Theorem 4.5.

**Theorem 4.6** If  $G$  is a  $(n, m)$  graph, then

$$E_{KCD_1}(G) \geq \frac{2Q + n|\beta_1||\beta_n|}{|\beta_1| + |\beta_n|}$$

where  $|\beta_1|$  is maximum and  $|\beta_n|$  is minimum of the absolute value of  $\beta_i$ 's.

**Proof.** Let  $a_i = 1$ ,  $b_i = |\beta_i|$ ,  $c_1 = |\beta_n|$  and  $c_2 = |\beta_1|$  in inequality (2.5)

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then,

$$\sum_{i=1}^n |\beta_i|^2 + |\beta_1||\beta_n| \sum_{i=1}^n 1^2 \leq (|\beta_1| + |\beta_n|)(\sum_{i=1}^n |\beta_i|) \quad (4.9)$$

Since,

$$E_{KCD_1(G)} = \sum_{i=1}^n |\beta_i| \quad \text{and} \quad \sum_{i=1}^n |\beta_i|^2 = 2Q.$$

Simplification of inequality (4.9) is

$$2Q + n|\beta_1||\beta_n| \leq (|\beta_1| + |\beta_n|)E_{KCD_1(G)}. \quad (4.10)$$

Simple calculation of inequality (4.10) yields the required result. □

**Illustration 4.6** Consider the graph  $H$  in the Figure 1, it has  $|\beta_1| = 30$ ,  $|\beta_4| = 10$ ,  $2Q = 1200$ ,  $n = 4$ . Further  $E_{KCD_1(H)} = 60$ . Hence it satisfies the Theorem 4.6.

**Theorem 4.7** If  $G$  is a  $r$ -regular graph, then the first KCD eigenvalues of  $G$  are  $-2(2r - 1)$  and  $2(n - 1)(2r - 1)$  with multiplicities  $(n-1)$  and  $1$  respectively and  $E_{KCD_1(G)} = 4(n - 1)(2r - 1)$ .

**Proof.**

$$\begin{aligned} |\beta I - KCD_1(G)| &= \begin{vmatrix} \beta & -2(2r-1) & -2(2r-1) & \cdots & -2(2r-1) \\ -2(2r-1) & \beta & -2(2r-1) & \cdots & -2(2r-1) \\ -2(2r-1) & -2(2r-1) & \beta & \cdots & -2(2r-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -2(2r-1) & -2(2r-1) & -2(2r-1) & \cdots & \beta \end{vmatrix} \\ &= (\beta + 2(2r-1))^{n-1} \begin{vmatrix} \beta & -2(2r-1) & -2(2r-1) & \cdots & -2(2r-1) \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \end{vmatrix} \\ &= (\beta - (n-1)2(2r-1))(\beta + 2(2r-1))^{n-1}. \end{aligned}$$

Hence,

$$E_{KCD_1}(G) = 4(n - 1)(2r - 1).$$

□

**Illustration 4.7** For the regular graph  $H$  in the Figure 1, the first  $KCD$  eigenvalues of  $H$  are  $-10$  (3 times) and  $30$  (1 time). Thus  $E_{KCD_1}(H) = 60$ .

## 5. Conclusion

In this article, we have introduced a new graph matrix called first  $KCD$  matrix and its related energy. Further, the computation of first  $KCD$  polynomials of some graphs has added a depth to this concept. The work is extended with the calculation of bounds for the largest first  $KCD$  eigenvalue and first  $KCD$  energy of a graph.

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## References

- [1] Adiga C., and Smitha M., (2009), On maximum degree energy of a graph, *Int. J. Contemp. Math. Sci.*, 4(8), 385-396.
- [2] Bernard S., and Child J. M., (2001), *Higher Algebra*, Macmillan India Ltd., New Delhi.
- [3] Biernacki M., Pidek H., and Ryll-Nardzewsk C., (1950), Sur une iné galité entre des intégrales définies, *Maria Curie SkÅĆodowska Univ.*, A(4), 1-4.
- [4] Chartrand G. and Lesniak L., (2000), *Graphs and Digraphs*, Chapman and Hall/ CRC, Washington, D. C.
- [5] Cvetković D., Doob M. and Sachs H., (1980), *Spectra of Graphs: Theory and Applications*, Academic Press, New York.
- [6] Diaz J. B. and Metcalf F. T., (1963), Stronger forms of a class of inequalities of G. Póly-G. Szegő and L. V. Kantorovich, *Bull. Amer. Math. Soc.*, 69(3), 415-418.
- [7] Gallian J. A., (2018), A Dynamic Survey of Graph Labeling, *Electron. J. Combin.*, #DS6.

- [8] Gutman I., (1978), The energy of a graph, Ber. Math. Stat. Sect. Forschungsz. Graz., 103, 1-22.
- [9] Harary F., (1969), Graph Theory, Addison-Wesely, Reading, Mass.
- [10] Mirajkar K. G. and Doddamani B. R., (2019), Bounds for the eigenvalues and energy of degree product adjacency matrix of a graph, J. Comput. Math. Sci., 10(3), 565-573.
- [11] Narayana N. S., (2016), Tutte polynomial of generalized flower graphs, Int. J. Math. Combin., 2, 29-42.
- [12] Ozeki N., (1968), On the estimation of inequalities by maximum and minimum values, J. College Arts and Sci., Chiba Univ., 5, 199-203.
- [13] Ramane H. S., Revankar D. S. and Patil J. B., (2013) Bounds for the degree sum eigenvalues and degree sum energy of a graph, Int. J. Pure Appl. Math. Sci., 6, 161-167.
- [14] Ramane H. S. and Shinde S. S., (2017), Degree exponent polynomial of graphs obtained by some graph operations, Electron. Notes Discrete Math., 63, 161-168.