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# MULTICRITERIA DECISION MAKING BASED ON SOME GENERALIZED POWER GEOMETRIC OPERATORS UNDER THE INTERVAL VALUED INTUITIONISTIC HESITANT FUZZY ENVIRONMENT

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**Abstract**

*The interval-valued intuitionistic hesitant fuzzy set, which is an extension of the fuzzy set, allows the membership of an element to a set of several possible interval-valued intuitionistic fuzzy numbers. So it is a very useful tool for modeling real life decision making problems. In this paper, we develop a series of generalized interval-valued intuitionistic hesitant fuzzy power geometric aggregation operators. Then, some desired properties of these aggregation operators are discussed. Furthermore, an approach to multicriteria decision making based on the interval-valued intuitionistic hesitant fuzzy power geometric operator is developed. Finally, a numerical example is provided to illustrate the proposed approach.*

**Keywords:** Interval-valued intuitionistic hesitant fuzzy set; Power aggregation operators; Multicriteria decision making .

**Mathematics Subject Classification** (2010) : 03E72, 90B50, 62C86.

## 1 Introduction

Most of the real life decision making problems involve imprecise or imperfect information. To deal with such imprecise information, fuzzy set [1] was introduced by Zadeh in 1965. A fuzzy set is characterized by a membership function which represents the degree of acceptance in a decision making problem. In real situation, however, there may be a hesitancy or uncertainty about the membership degree of the object in that set. So, as its consequence, Atanassov [2, 3] introduced the intuitionistic fuzzy sets (IFSs) in 1983 that is characterized by the degrees of membership and non-membership. In the case of IFS, the non-membership grade expresses the degree of rejection in a decision making problem. Later, Atanassov and Gargov [4] introduced the interval-valued intuitionistic fuzzy set (IVIFS) as a further generalization of IFS in which intervals in $[0,1]$ are used for membership and non-membership values rather than exact numerical values. Xu et al. [13] developed a series of aggregation operators under the interval-valued intuitionistic fuzzy environment such as the interval-valued intuitionistic fuzzy weighted arithmetic aggregation (IIFWA), the interval-valued intuitionistic fuzzy ordered weighted aggregation (IIFOWA) and the interval-valued intuitionistic fuzzy hybrid aggregation (IIFHA) operator. Xu et al. [15] further proposed the interval-valued intuitionistic fuzzy weighted geometric (IIFWG) operator, the interval-valued intuitionistic fuzzy ordered weighted geometric (IIFOWG) operator and the interval-valued intuitionistic fuzzy hybrid geometric (IIFHG) operator. It is observed that, in fuzzy multicriteria decision making (MCDM) problems, representation of membership degrees of objects to a certain set is not unique. To deal with such type of difficulty, Torra and Narukawa [8] and Torra [9] defined a hesitant fuzzy set (HFS) as an extension of the fuzzy set (FS). Chen et al. [11] extended this to include interval-valued hesitant fuzzy set (IVHFS) in which the membership degrees of an element to a given set are not exactly defined but denoted by several possible interval values. Further, Zhang [12] proposed the concept of the interval-valued intuitionistic hesitant fuzzy set (IVIHFS) and developed several series of aggregation operators under the interval-valued intuitionistic hesitant fuzzy environment. The interval-valued intuitionistic hesitant fuzzy set allows the membership of an element to be a set of several possible interval-valued intuitionistic fuzzy numbers.

Most of the above mentioned aggregation operators do not consider the information about the relationship between the values being fused. Yager [16] proposed the power average

(PA) operator and the power ordered weighted average (POWA) operator, for which the weighting vectors depend on the input arguments. It is observed that the PA and POWA operators allow the values being aggregated to support and reinforce each other. Xu and Yager [17] proposed the power geometric (PG) operator and and the power ordered weighted geometric (POWG) operator. Zhou and Chen [19] proposed the generalized power average (GPA) operator. Xu [18] proposed several atanassov's intuitionistic fuzzy power geometric aggregation operators. Zhang [20] proposed a series of generalized atanassov's intuitionistic fuzzy power geometric aggregation operators. Further, He et al. [21] introduced a series of generalized interval-valued atanassov's intuitionistic fuzzy power aggregation operators. Zhang [22] developed a series of hesitant fuzzy power aggregation operators.

In some real life decision making problems, it is difficult to specify the precise membership degrees of an element to a set. Since, the interval-valued intuitionistic hesitant fuzzy set (IVIHFS) proposed by Zhang [12] allows the membership degrees of an element to be a set of several possible interval-valued intuitionistic fuzzy numbers (IVIFNs), so the interval-valued intuitionistic hesitant fuzzy set (IVIHFS) is a very useful tool to express decision maker's (DM's) hesitancy among several possible interval-valued intuitionistic fuzzy numbers (IVIFNs).

Motivated by the power average operators (PA) [16, 17], in this paper, a series of generalized interval-valued intuitionistic hesitant fuzzy power average operators are developed for which the weighting vectors depend upon the input arguments and thus allow the values being aggregated to support and reinforce each other. The main characteristic of these operators is that they not only accommodate situations in which the input arguments are IVIHFNs, but they also consider information about the relationship between the IVIHFNs being fused.

The rest of the paper proceeds as follows. Section 2 briefly reviews some basic concepts, Section 3 Proposes the generalized interval- valued intuitionistic hesitant fuzzy power geometric operator. Section 4 presents an approach to multicriteria decision making based on the proposed operator. Section 5 shows the feasibility and validity of the approach to multicriteria decision making by a numerical example. Section 6 provides the concluding remarks.

## 2 Preliminaries

We compile in this section the relevant notion required for the development of the present paper.

Atanassov [2] introduced the concept of the intuitionistic fuzzy set (IFS) as follows:

**Definition 2.1** [2] Let X be a finite non empty set, then an intuitionistic fuzzy set (IFS) on X is an object A given by

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\},$$

where the functions $\mu_A: X \to [0,1]$ and $\nu_A: X \to [0,1]$ define the degree of membership and the degree of non-membership of the element $x \in X$ to $A$ together with the condition that $0 \leqslant \mu_A(x) + \nu_A(x) \leqslant 1$ for all $x \in X$. Furthermore, $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ for all $x \in X$ is called the hesitancy degree or intuitionistic index of $x \in X$ to $A$. If $\pi_A(x) = 0$, i.e., $\mu_A(x) + \nu_A(x) = 1$, then intuitionistic fuzzy set (IFS) $A$ reduces to a fuzzy set [1].

Xu et al. [5] called each pair $(\mu_A(x), \nu_A(x))$ an intuitionistic fuzzy number (IFN), and for convenience, each IFN is denoted by $\alpha = (\mu_\alpha, \nu_\alpha)$, where $\mu_\alpha, \nu_\alpha \geqslant 0$ and $\mu_\alpha + \nu_\alpha \leqslant 1$

Atanassov [4] further proposed the interval-valued intuitionistic fuzzy set (IVIFS) as follows:

**Definition 2.2** [4] Let X be a finite non-empty set, then an interval-valued intuitionistic fuzzy set (IVIFS) on $X$ is an object $\tilde{A}$ given by

$$\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)\rangle | x \in X\},$$

where, $\mu_{\tilde{A}}(x)$ and $\nu_{\tilde{A}}(x)$ respectively, called the membership degree and the non-membership degree of the element $x \in X$ to $\tilde{A}$, satisfying $\mu_{\tilde{A}}(x) \subset [0,1]$ and $\nu_{\tilde{A}}(x) \subset [0,1]$.

Xu [6] called each pair $(\mu_{\tilde{A}}, \nu_{\tilde{A}})$ an interval-valued intuitionistic fuzzy number (IVIFN), where $\mu_{\tilde{A}} = [\mu_{\tilde{A}}^-, \mu_{\tilde{A}}^+]$ and $\nu_{\tilde{A}} = [\nu_{\tilde{A}}^-, \nu_{\tilde{A}}^+]$ are interval numbers, with the condition $0 \leqslant \mu_{\tilde{A}}^+ + \nu_{\tilde{A}}^+ \leqslant 1$.

Torra [9] introduced the concept of the hesitant fuzzy set (HFS) as a generalization of fuzzy set [1] which is defined as follows:

**Definition 2.3** [9*] Let $X$ be a fixed set, then a hesitant fuzzy set on $X$ is defined in terms of a function that when applied to $X$ returns a subset of* $[0,1]$.

*Xia and Xu [10] expressed the HFS as follows:*

$$E = \{\langle x, h_E(x)\rangle | x \in X\}$$

where $h_E(x)$ is a set of some values in $[0,1]$, which denotes the possible membership degree of the element $x \in X$ to the set $E$.

Zhang [12] proposed the concept of interval-valued intuitionistic hesitant fuzzy set, which extends the hesitant fuzzy set to interval-valued intuitionistic fuzzy environments.

**Definition 2.4** *[12] Let $X$ be a fixed set, an interval-valued intuitionistic hesitant fuzzy set (IVIHFS) on $X$ is defined in terms of a function that when applied to $X$ returns a subset of $\Omega$. The interval-valued intuitionistic hesitant fuzzy set (IVIHFS) is expressed as follows:*

$$\tilde{E} = \{\langle x, h_{\tilde{E}}(x)\rangle | x \in X\}$$

where $h_{\tilde{E}}(x)$ is a set of some interval-valued intuitionistic hesitant fuzzy numbers (IVIFNs) in $X$, which denotes the possible membership degree of the element $x \in X$ to the set $\tilde{E}$.

Zhang [12] called $\tilde{h} = h_{\tilde{E}}(x)$ an interval-valued intuitionistic hesitant fuzzy element (IVIHFE). If $\alpha \in \tilde{h}$, then $\alpha$ is an IVIFN, denoted by $\alpha = (\mu_\alpha, \nu_\alpha) = ([\mu_\alpha^-, \mu_\alpha^+], [\nu_\alpha^-, \nu_\alpha^+])$. For any $\alpha \in \tilde{h}$, if $\alpha \in [0,1]$, $\tilde{h}$ reduces to hesitant fuzzy element (HFE) [10], if $\alpha$ is a closed sub-interval of the unit interval, then $\tilde{h}$ reduces to an interval-valued hesitant fuzzy element (IVHFE) [11], if $\alpha$ is an intuitionistic fuzzy number (IFN) [5], then $\tilde{h}$ reduces to an intuitionistic hesitant fuzzy element (IHFE). Hence, HFEs, IVHFEs, and IHFEs are special cases of IVIHFEs.

Zhang [12] defined some operations on interval-valued intuitionistic hesitant fuzzy elements (IVIHFEs) as follows:

**Definition 2.5** *[12] Given three IVIHFEs $\tilde{h}, \tilde{h}_1, \tilde{h}_2$ and a scalar $\lambda > 0$, we the following operations:*

(i) $\tilde{h}^c = \{\alpha^c | \alpha \in \tilde{h}\} = \{([\nu_\alpha^-, \nu_\alpha^+], [\mu_\alpha^-, \nu_\alpha^+]) | \alpha \in \tilde{h}\}$

(ii) $\tilde{h}_1 \cup \tilde{h}_2 = \{\max(\alpha_1, \alpha_2) | \alpha_1 \in \tilde{h}_1, \alpha_2 \in \tilde{h}_2\}$
$= \{[\max(\mu_{\alpha_1}^-, \mu_{\alpha_2}^-), \max(\mu_{\alpha_1}^+, \mu_{\alpha_2}^+)], [\min(\nu_{\alpha_1}^-, \nu_{\alpha_2}^-), \min(\nu_{\alpha_1}^+, \nu_{\alpha_2}^+)] | \alpha_1 \in \tilde{h}_1, \alpha_2 \in \tilde{h}_2\}$

(iii) $\tilde{h}_1 \cap \tilde{h}_2 = \{\min(\alpha_1, \alpha_2) | \alpha_1 \in \tilde{h}_1, \alpha_2 \in \tilde{h}_2\}$
$= \{[\min(\mu_{\alpha_1}^-, \mu_{\alpha_2}^-), \min(\mu_{\alpha_1}^+, \mu_{\alpha_2}^+)], [\max(\nu_{\alpha_1}^-, \nu_{\alpha_2}^-), \max(\nu_{\alpha_1}^+, \nu_{\alpha_2}^+)] | \alpha_1 \in \tilde{h}_1, \alpha_2 \in \tilde{h}_2\}$

(iv) $\tilde{h}_1 \oplus \tilde{h}_2 = \{[\mu_{\alpha_1}^- + \mu_{\alpha_2}^- - \mu_{\alpha_1}^- \mu_{\alpha_2}^-, \mu_{\alpha_1}^+ + \mu_{\alpha_2}^+ - \mu_{\alpha_1}^+ \mu_{\alpha_2}^+], [\nu_{\alpha_1}^- \nu_{\alpha_2}^-, \nu_{\alpha_1}^+ \nu_{\alpha_2}^+] | \alpha_1 \in \tilde{h}_1, \alpha_2 \in \tilde{h}_2\}$

(v) $\tilde{h}_1 \otimes \tilde{h}_2 = \{[\mu_{\alpha_1}^- \mu_{\alpha_2}^-, \mu_{\alpha_1}^+ \mu_{\alpha_2}^+], [\nu_{\alpha_1}^- + \nu_{\alpha_2}^- - \nu_{\alpha_1}^- \nu_{\alpha_2}^-, \nu_{\alpha_1}^+ + \nu_{\alpha_2}^+ - \nu_{\alpha_1}^+ \nu_{\alpha_2}^+] | \alpha_1 \in \tilde{h}_1, \alpha_2 \in \tilde{h}_2\}$

(vi) $\lambda\tilde{h} = \{\lambda\alpha | \alpha \in \tilde{h}\} = \{([1-(1-\mu_\alpha^-)^\lambda, 1-(1-\mu_\alpha^+)^\lambda], [(\nu_\alpha^-)^\lambda, (\nu_\alpha^+)^\lambda]) | \alpha \in \tilde{h}\}$

(vii) $\tilde{h}^\lambda = \{\alpha^\lambda | \alpha \in \tilde{h}\} = \{([(\mu_\alpha^-)^\lambda, (\mu_\alpha^+)^\lambda], [1-(1-\nu_\alpha^-)^\lambda, 1-(1-\nu_\alpha^+)^\lambda]) | \alpha \in \tilde{h}\}$

From the above defined operations it is obvious that $\tilde{h}_1 \oplus \tilde{h}_2, \tilde{h}_1 \otimes \tilde{h}_2, \lambda\tilde{h}$, and $\tilde{h}^\lambda$ are IVIHFEs.

Zhang [12] further established the following operational laws on the IVIHFEs as follows:

**Definition 2.6** *[12] Let $\tilde{h}, \tilde{h}_1$, and $\tilde{h}_2$ be three IVIHFEs, and scalars $\lambda, \lambda_1$, and $\lambda_2 > 0$, then the following operation law hold:*

[R1] $\tilde{h}_1^c \cup \tilde{h}_2^c = (\tilde{h}_1 \cap \tilde{h}_2)^c$
[R2] $\tilde{h}_1^c \cap \tilde{h}_2^c = (\tilde{h}_1 \cup \tilde{h}_2)^c$
[R3] $(\tilde{h}^c)^\lambda = (\lambda\tilde{h})^c$
[R4] $\lambda(\tilde{h}^c) = (\tilde{h}^\lambda)^c$
[R5] $\tilde{h}_1^c \oplus \tilde{h}_2^c = (\tilde{h}_1 \otimes \tilde{h}_2)^c$
[R6] $\tilde{h}_1^c \otimes \tilde{h}_2^c = (\tilde{h}_1 \oplus \tilde{h}_2)^c$
[R7] $\tilde{h}_1 \oplus \tilde{h}_2 = \tilde{h}_2 \oplus \tilde{h}_1$
[R8] $\lambda(\tilde{h}_1 \oplus \tilde{h}_2) = \lambda\tilde{h}_1 \oplus \lambda\tilde{h}_2$
[R9] $(\lambda_1\lambda_2)\tilde{h} = \lambda_1(\lambda_2\tilde{h})\tilde{h}_1 \otimes \tilde{h}_2 = \tilde{h}_2 \otimes \tilde{h}_1$
[R10] $\tilde{h}_1^{\lambda_1} \otimes \tilde{h}_2^{\lambda_2} = (\tilde{h}_1 \otimes \tilde{h}_2)^\lambda$
[R11] $\tilde{h}^{\lambda_1\lambda_2} = (\tilde{h}^{\lambda_1})^{\lambda_2}$

**Definition 2.7** [12] Let$\tilde{h} = \{([\mu_\alpha^-, \mu_\alpha^+], [\nu_\alpha^-, \nu_\alpha^+]) | \alpha \in \tilde{h}\}$be an IVIHFE, then the Score function S(h̃), and the Accuracy function of $A(\tilde{h})$are defined as follows:

$$S(\tilde{h}) = \frac{\sum_{\alpha \in \tilde{h}} S(\alpha)}{\#\tilde{h}}, (2.1)$$

$$A(\tilde{h}) = \frac{\sum_{\alpha \in \tilde{h}} A(\alpha)}{\#\tilde{h}} (2.2)$$

where, $\#\tilde{h}$ denotes the number of the elements in $\tilde{h}$, and $S(\alpha)$ and $A(\alpha)$ represent the Score and the Accuracy function of the IVIFEs (Xu [6]) which are given as:

$$S(\alpha) = \frac{\mu_\alpha^- - \nu_\alpha^- + \mu_\alpha^+ - \nu_\alpha^+}{2}, (2.3)$$

and

$$A(\alpha) = \frac{\mu_\alpha^- + \nu_\alpha^- + \mu_\alpha^+ + \nu_\alpha^+}{2}. (2.4)$$

Also for any two IVIHFEs $\tilde{h}_1$ and $\tilde{h}_2$, the following rules are hold:

**(R 1)** if $S(\tilde{h}_1) > S(\tilde{h}_2)$, then $\tilde{h}_1 \succ \tilde{h}_2$

**(R 2)** if $S(\tilde{h}_1) < S(\tilde{h}_2)$, then $\tilde{h}_1 \prec \tilde{h}_2$

**(R 3)** if $S(\tilde{h}_1) = S(\tilde{h}_2)$, then the following hold,

(a) if $A(\tilde{h}_1) > A(\tilde{h}_2)$, then $\tilde{h}_1 \succ \tilde{h}_2$

(b) if $A(\tilde{h}_1) < A(\tilde{h}_2)$, then $\tilde{h}_1 \prec \tilde{h}_2$

(c) if $A(\tilde{h}_1) = A(\tilde{h}_2)$, then $\tilde{h}_1 = \tilde{h}_2$.

## 2.1 Power Aggregation Operators

In this subsection, we briefly review some of the existing power aggregation operators.

Yager [16] introduced the concept of the power average (PA) operator as follows:

**Definition 2.8** [16] The power average (PA) operator is a mapping PA: $\mathbb{R}^n \rightarrow \mathbb{R}$, which is defined as follows:

$$\mathrm{PA}(a_1, a_2, \ldots, a_n) = \frac{\sum_{i=1}^{n}(1+T(a_i))a_i}{\sum_{i=1}^{n}(1+T(a_i))} \quad (2.5)$$

where

$$T(a_i) = \sum_{\substack{j=1 \\ j\neq i}}^{n} Supp(a_i, a_j) \quad (2.6)$$

and $Supp(a_i, a_j)$ is the support of $a_i$ from $a_j$, which satisfies the following properties.

(i) $Supp(a_i, a_j) \in [0,1]$,

(ii) $Supp(a_i, a_j) = Supp(a_j, a_i)$,

(iii) If $|a_i - a_j| < |a_s - a_t|$, then $Supp(a_i, a_j) \geq Supp(a_s, a_t)$.

Xu and Yager [17] further proposed the power geometric (PG) operator as follows:

**Definition 2.9** [17] The power geometric (PG) operator is a mapping PG: $\mathbb{R}^n \rightarrow \mathbb{R}$, which is defined as follows:

$$\mathrm{PG}(a_1, a_2, \ldots, a_n) = \prod_{i=1}^{n} a_i^{\frac{1+T(a_i)}{\sum_{i=1}^{n}(1+T(a_i))}} \quad (2.7)$$

where

$$T(a_i) = \sum_{\substack{j=1 \\ j\neq i}}^{n} Supp(a_i, a_j) \quad (2.8)$$

and $Supp(a_i, a_j)$ is the support of $a_i$ from $a_j$, which satisfies the three properties mention in Definition 2.8.

The power average (PA) and the power geometric (PG) operators are a nonlinear weighted aggregation operators, whose weighting vectors depend upon the input data and allow values being aggregated to support and reinforce each other. The closer two values $a_i$ and $a_j$, the more similar they are, and the more they support each other.

**Definition 2.10** [16] The power ordered weighted average (POWA) operator is a mappingPOWA: $\mathbb{R}^n \to \mathbb{R}$, which is defined as follows :

$$\mathrm{POWA}(a_1, a_2, \dots, a_n) = \sum_{i=1}^{n} u_i a_{\mathrm{index}(i)} \quad (2.9)$$

where

$$u_i = g(\frac{R_i}{TV}) - g(\frac{R_{i-1}}{TV}), \quad R_i = \sum_{j=1}^{i} V_{\mathrm{index}(j)}, \quad TV = \sum_{i=1}^{n} V_{\mathrm{index}(i)},$$

$$V_{\mathrm{index}(i)} = 1 + T(a_{\mathrm{index}(i)}),$$

$$T(a_{\mathrm{index}(i)}) = \sum_{\substack{j=1 \\ j\neq i}}^{n} Supp(a_{\mathrm{index}(i)}, a_{\mathrm{index}(j)}) \quad (2.10)$$

In the above equations, $a_{\mathrm{index}(i)}$ is the $i$th largest argument among all of the arguments $a_j (j = 1,2, \dots, n)$, $T(a_{\mathrm{index}(i)})$ denotes the support of the $i$th largest argument by all of the other arguments, and $Supp(a_{\mathrm{index}(i)}, a_{\mathrm{index}(j)})$ indicates the support of $j$th largest argument for the $i$th largest argument. Also, $g: [0,1] \to [0,1]$ is a basic unit-interval monotonic (BUM) function that has the following properties

(i) $g(0) = 0$ (ii) $g(1) = 1$ (iii) $g(x) \geq g(y)$ if $x > y$

**Definition 2.11** [17] The power ordered weighted geometric (POWG) operator is a mapping

POWG: $\mathbb{R}^n \to \mathbb{R}$, which is defined as follows:

$$\text{POWG}(a_1, a_2, \dots, a_n) = \prod_{i=1}^{n} a_{\text{index}(i)}^{u_i} \quad (2.11)$$

where $u_i$ is given by Equation (2.10) as defined in Definition 2.10 and $a_{\text{index}(i)}$ is the $i$th largest argument among all of the arguments $a_j (j = 1,2, \dots, n)$.

Based on the power average (PA) operator [16], Xu[18] proposed a series of intuitionistic fuzzy power aggregation operators.

**Definition 2.12** [18] Let $\alpha_i = (\mu_{\alpha_i}, \nu_{\alpha_i})$ $(i = 1,2, \dots, n)$ be a collection of intuitionistic fuzzy numbers (IFNs), and $w = (w_1, w_2, \dots, w_n)^T$ be the weight vector of $\alpha_i$ $(i = 1,2, \dots, n)$, *where* $w_i \geqslant 0$ $(i = 1,2, \dots, n)$ *and* $\sum_{i=1}^{n} w_i = 1$, then the intuitionistic fuzzy power weighted average (IFPWA) operator is given by

$$\text{IFPWA}(\alpha_1, \alpha_2, \dots, \alpha_n) = \frac{\bigoplus_{j=1}^{n} w_j((1+T(\alpha_j))\alpha_j)}{\sum_{i=1}^{n} w_i(1+T(\alpha_i))}$$

$$= \left(1 - \prod_{j=1}^{n} (1 - \mu_{\alpha_j})^{\frac{w_j(1+T(\alpha_j))}{\sum_{i=1}^{n} w_i(1+T(\alpha_i))}}, \prod_{j=1}^{n} (\nu_{\alpha_j})^{\frac{w_j(1+T(\alpha_j))}{\sum_{i=1}^{n} w_i(1+T(\alpha_i))}}\right) \quad (2.12)$$

where, $T(\alpha_i) = \sum_{\substack{j=1 \\ j\neq 1}}^{n} w_j Supp(\alpha_i, \alpha_j)$ with the conditions $w_j \in [0,1], (j = 1,2, \dots, n)$ and $\sum_{j=1}^{n} w_j = 1$ and $Supp(\alpha_i, \alpha_j)$ denotes the support of $\alpha_i$ from $\alpha_j$ and satisfies the following three properties,

(i) $Supp(\alpha_i, \alpha_j) \in [0,1]$,

(ii) $Supp(\alpha_i, \alpha_j) = Supp(\alpha_j, \alpha_i)$,

(iii) If $d(\alpha_i, \alpha_j) < d(\alpha_s, \alpha_t)$, then $Supp(\alpha_i, \alpha_j) \geqslant Supp(\alpha_s, \alpha_t)$, where d is a distance measure between two IFNs (such as the normalized Hamming [7]).

**Definition 2.13** [18] Let $\alpha_i = (\mu_{\alpha_i}, \nu_{\alpha_i})$ $(i = 1,2, \dots, n)$ be a collection of intuitionistic fuzzy numbers (IFNs), then the intuitionistic fuzzy power ordered weighted average (IFPOWA) operator is defined as follows:

$$\text{IFPOWA}(\alpha_1, \alpha_2, \dots, \alpha_n) = \bigoplus_{i=1}^{n} u_i \alpha_{\text{index}(i)}$$

$$= \left(1 - \prod_{i=1}^{n} (1 - \mu_{\alpha_{\text{index}(i)}})^{u_i}, \prod_{i=1}^{n} (\nu_{\alpha_{\text{index}(i)}})^{u_i}\right) \quad (2.13)$$

where

$$u_i = g\left(\frac{R_i}{TV}\right) - g\left(\frac{R_{i-1}}{TV}\right), \quad R_i = \sum_{j=1}^{i} V_{\text{index}(j)}, \quad TV = \sum_{i=1}^{n} V_{\text{index}(i)},$$

$$V_{\text{index}(i)} = 1 + T(\alpha_{\text{index}(i)}),$$

$$T(\alpha_{\text{index}(i)}) = \sum_{\substack{j=1 \\ j \neq i}}^{n} Supp(\alpha_{\text{index}(i)}, \alpha_{\text{index}(j)}). \quad (2.14)$$

In Equation (2.14), $\alpha_{\text{index}(i)}$ is the $i$th largest IFN of all the IFNs $\alpha_j (j = 1,2, \dots, n)$, $T(\alpha_{\text{index}(i)})$ denotes the support of the $i$th largest IFN by all the other IFNs, and $Supp(\alpha_{\text{index}(i)}, \alpha_{\text{index}(j)})$ denotes the support of the $j$th largest IFN for the $i$th largest IFN. Also, the function $g: [0,1] \to [0,1]$ is a basic unit-interval monotonic (BUM) function which satisfies the following conditions :

(i) $g(0) = 0$

(ii) $g(1) = 1$

(iii) if $x > y$, then $g(x) \geqslant g(y)$

Futher, Zhang [20] proposed a series of generalized Atanassov's intuitionistic fuzzy power geometric operators.

**Definition 2.14** [20] Let$\alpha_i = (\mu_{\alpha_i}, \nu_{\alpha_i})$ $(i = 1,2, \ldots, n)$be a collection of IFNs, and$\lambda > 0$. A weighted generalized Atanassov's intuitionistic fuzzy power average (WGIFPA) operator is defined as follows:

$$\mathrm{WGIFPA} = \left( \frac{\bigoplus_{j=1}^{n} w_j(1+T(\alpha_j))\alpha_j^{\lambda}}{\sum_{i=1}^{n} w_i(1+T(\alpha_i))} \right)^{\frac{1}{\lambda}} \quad (2.15)$$

where $T(\alpha_i) = \sum_{\substack{j=1 \\ j\neq 1}}^{n} w_j Supp(\alpha_i, \alpha_j)$ with the conditions $w_j \in [0,1], (j = 1,2, \ldots, n)$ and $\sum_{j=1}^{n} w_j = 1$. $Supp(\alpha_i, \alpha_j)$ denotes the support of $\alpha_i$ from $\alpha_j$ and satisfies the three conditions mentioned in Definition 2.12.

**Definition 2.15** [20] Let$\alpha_i = (\mu_{\alpha_i}, \nu_{\alpha_i})$ $(i = 1,2, \ldots, n)$be a collection of IFNs, and$\lambda > 0$. A generalized Atanassov's intuitionistic fuzzy power geometric averaging (GIFPGA) operator is defined as follows:

$$\mathrm{GIFPGA}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \frac{1}{\lambda}\left(\bigotimes_{i=1}^{n} (\lambda\alpha_i)^{\frac{1+T(\alpha_i)}{\sum_{i=1}^{n}(1+T(\alpha_i))}}\right) (2.16)$$

where $T(\alpha_i)$ is as given in Definition 2.12.

He et al. [21] extended the Atanassov's intuitionistic fuzzy power aggregation operators to develop a series of Atanassov's interval-valued intuitionistic fuzzy power aggregation operators.

**Definition 2.16**[21] Let$\tilde{A}_i = \langle [\mu_{\tilde{A}_i}^-, \mu_{\tilde{A}_i}^+], [\nu_{\tilde{A}_i}^-, \nu_{\tilde{A}_i}^+] \rangle$ $(i = 1,2,\ldots,n)$ be a collection of IVIFNs. Then the interval-valued intuitionistic fuzzy power average (IVIFPA) operator is defined as,

$$\text{IVIFPA}(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n) = \frac{\bigoplus_{j=1}^{n} (1+T(\tilde{A}_j))\tilde{A}_j}{\sum_{i=1}^{n} (1+T(\tilde{A}_i))}$$

$$= \left\langle \left[1 - \prod_{i=1}^{n} (1 - \mu_{\tilde{A}_j}^-)^{\frac{(1+T(\tilde{A}_j))}{\sum_{i=1}^{n}(1+T(\tilde{A}_i))}}, 1 - \prod_{i=1}^{n} (1 - \mu_{\tilde{A}_j}^+)^{\frac{(1+T(\tilde{A}_j))}{\sum_{i=1}^{n}(1+T(\tilde{A}_i))}}\right]\right.$$

$$\left.\left[\prod_{j=1}^{n} (\nu_{\tilde{A}_j}^-)^{\frac{(1+T(\tilde{A}_j))}{\sum_{i=1}^{n}(1+T(\tilde{A}_i))}}, \prod_{j=1}^{n} (\nu_{\tilde{A}_j}^+)^{\frac{(1+T(\tilde{A}_j))}{\sum_{i=1}^{n}(1+T(\tilde{A}_i))}}\right]\right\rangle (2.17)$$

where

$$T(\tilde{A}_i) = \sum_{\substack{j=1 \\ j \neq i}}^{n} Supp(\tilde{A}_i, \tilde{A}_j) (2.18)$$

$Supp(\tilde{A}_i, \tilde{A}_j)$ denotes the support of $\tilde{A}_i$ from $\tilde{A}_j$, which satisfies :

(i) $Supp(\tilde{A}_i, \tilde{A}_j) \in [0,1]$

(ii) $Supp(\tilde{A}_i, \tilde{A}_j) = Supp(\tilde{A}_j, \tilde{A}_i)$

(iii) $Supp(\tilde{A}_i, \tilde{A}_j) \geqslant Supp(\tilde{A}_r, \tilde{A}_s)$, if $d(\tilde{A}_i, \tilde{A}_j) < d(\tilde{A}_r, \tilde{A}_s)$, where $d$ is is the distance between $\tilde{A}_i$ and $\tilde{A}_j$.

**Definition 2.17** [21] Let$\tilde{A}_i = \langle [\mu_{\tilde{A}_i}^-, \mu_{\tilde{A}_i}^+], [\nu_{\tilde{A}_i}^-, \nu_{\tilde{A}_i}^+] \rangle$ $(i = 1,2,\ldots,n)$be a collection of IVIFNs. Then the interval-valued intuitionistic fuzzy power ordered weighted average (IVIFPOWA) operator is defined as,

$$\text{IVIFPOWA} = \bigoplus_{i=1}^{n} u_i \tilde{A}_{\text{index}(i)} \quad (2.19)$$

where

$$u_i = g\left(\frac{R_i}{TV}\right) - g\left(\frac{R_{i-1}}{TV}\right), \quad R_i = \sum_{j=1}^{i} V_{\text{index}(j)}, \quad TV = \sum_{i=1}^{n} V_{\text{index}(i)},$$

$$V_{\text{index}(i)} = 1 + T(\tilde{A}_{\text{index}(i)}),$$

$$T(\tilde{A}_{\text{index}(i)}) = \sum_{\substack{j=1 \\ j\neq i}}^{n} Supp(\tilde{A}_{\text{index}(i)}, \tilde{A}_{\text{index}(j)}) \quad (2.20)$$

In Equation (2.20), $\alpha_{\text{index}(i)}$ is the $i$th largest IVIFN of all the IVIFNs $\alpha_j (j = 1,2, \dots, n)$, $T(\alpha_{\text{index}(i)})$ denotes the support of the $i$th largest IVIFN by all the other IVIFNs, and $Supp(\alpha_{\text{index}(i)}, \alpha_{\text{index}(j)})$ denotes the support of the $j$th largest IVIFN for the $i$th largest IVIFN. Also, the function $g: [0,1] \to [0,1]$ is a basic unit-interval monotonic (BUM) function which satisfies the following conditions:

(i) $g(0) = 0$
(ii) $g(1) = 1$
(iii) if $x > y$ for $x, y \in [0,1]$, then $g(x) \geqslant g(y)$

**Definition 2.18** [21] Let $\tilde{A}_i = \langle [\mu^-_{\tilde{A}_i}, \mu^+_{\tilde{A}_i}], [\nu^-_{\tilde{A}_i}, \nu^+_{\tilde{A}_i}] \rangle$ $(i = 1,2, \dots, n)$ be a collection of IVIFNs, and $\lambda > 0$. Then the generalized interval-valued intuitionistic fuzzy power averaging (GIVIFPA) operator is defined as,

$$\text{GIVIFPA}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n) = \frac{\bigoplus_{j=1}^{n} (1 + T(\tilde{A}_j)) \tilde{A}_j^{\lambda}}{\sum_{i=1}^{n} (1 + T(\tilde{A}_i))}$$

$$= \left\langle \begin{bmatrix} 1-\left(\prod_{j=1}^{n}(1-(\mu_{\tilde{A}_j}^{-})^{\lambda})^{\frac{(1+T(\tilde{A}_j))}{\sum_{i=1}^{n}(1+T(\tilde{A}_i))}}\right)^{\frac{1}{\lambda}}, \\ 1-\left(\prod_{j=1}^{n}(1-(\mu_{\tilde{A}_j}^{+})^{\lambda})^{\frac{(1+T(\tilde{A}_j))}{\sum_{i=1}^{n}(1+T(\tilde{A}_i))}}\right)^{\frac{1}{\lambda}} \end{bmatrix}, \begin{bmatrix} 1-\left(1-\prod_{j=1}^{n}\left(1-(1-\nu_{\tilde{A}_j}^{-})^{\lambda}\right)^{\frac{(1+T(\tilde{A}_j))}{\sum_{i=1}^{n}(1+T(\tilde{A}_i))}}\right)^{\frac{1}{\lambda}}, \\ 1-\left(1-\prod_{j=1}^{n}\left(1-(1-\nu_{\tilde{A}_j}^{+})^{\lambda}\right)^{\frac{(1+T(\tilde{A}_j))}{\sum_{i=1}^{n}(1+T(\tilde{A}_i))}}\right)^{\frac{1}{\lambda}} \end{bmatrix} \right\rangle \quad (2.21)$$

where $T(\tilde{A}_i)$ satisfies Equation (2.18).

**Definition 2.19** [21] Let $\tilde{A}_i = \langle[\mu_{\tilde{A}_i}^{-}, \mu_{\tilde{A}_i}^{+}], [\nu_{\tilde{A}_i}^{-}, \nu_{\tilde{A}_i}^{+}]\rangle$ $(i = 1,2,\dots,n)$be a collection of IVIFNs. Then the generalized interval-valued intuitionistic fuzzy power ordered weighted average (GIVIFPOWA) operator is defined as,

$$\text{GIVIFPOWA}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n) = (\bigoplus_{i=1}^{n} u_i \tilde{A}_{\text{index}(i)}^{\lambda})^{\frac{1}{\lambda}}$$

$$= \left\langle \begin{bmatrix} \left(1-\prod_{i=1}^{n}(1-(\mu_{\text{index}(i)}^{-})^{\lambda})^{u_i}\right)^{\frac{1}{\lambda}}, \\ \left(1-\prod_{i=1}^{n}(1-(\mu_{\text{index}(i)}^{+})^{\lambda})^{u_i}\right)^{\frac{1}{\lambda}} \end{bmatrix}, \begin{bmatrix} 1-\left(1-\prod_{i=1}^{n}(1-(1-\nu_{\text{index}(i)}^{-})^{\lambda})^{u_i}\right)^{\frac{1}{\lambda}}, \\ 1-\left(1-\prod_{i=1}^{n}(1-(1-\nu_{\text{index}(i)}^{+})^{\lambda})^{u_i}\right)^{\frac{1}{\lambda}} \end{bmatrix} \right\rangle \quad (2.22)$$

where $u_i$ satisfies Equation (2.20).

## 3 Some Generalized Interval-Valued Intuitionistic Hesitant Fuzzy Power Geometric Operators

In this section, we propose a series of generalized interval-valued intuitionistic hesitant fuzzy power geometric operators and investigate some of their desirable properties.

**Definition 3.1** *Let* $\tilde{h}_i$ $(i = 1,2,\dots,n)$ be a collection of interval valued intuitionistic hesitant fuzzy elements (IVIHFEs), and a scalar $\lambda \in (-\infty, +\infty), \lambda \neq 0$, then the generalized interval-valued intuitionistic hesitant fuzzy power geometric (GIVIHFPG) operator is a mapping GIVIHFPG: $\widetilde{H}^n \to \widetilde{H}$, such that

$$\mathrm{GIVIHFPG}(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_n) = \frac{1}{\lambda}\left(\bigotimes_{i=1}^{n} (\lambda \tilde{h}_i)^{(1+T(\tilde{h}_i))/\sum_{i=1}^{n}(1+T(\tilde{h}_i))}\right) \quad (3.1)$$

where

$$T(\tilde{h}_i) = \sum_{\substack{j=1 \\ j\neq i}}^{n} Supp(\tilde{h}_i, \tilde{h}_j) \quad (3.2)$$

$Supp(\tilde{h}_i, \tilde{h}_j)$ denotes the support of $\tilde{h}_i$ from $\tilde{h}_j$ and satisfies the following three properties,

(i) $Supp(\tilde{h}_i, \tilde{h}_j) \in [0,1]$,

(ii) $Supp(\tilde{h}_i, \tilde{h}_j) = Supp(\tilde{h}_j, \tilde{h}_i)$,

(iii) If $d(\tilde{h}_i, \tilde{h}_j) < d(\tilde{h}_s, \tilde{h}_t)$, then $Supp(\tilde{h}_i, \tilde{h}_j) \geqslant Supp(\tilde{h}_s, \tilde{h}_t)$, where $d$ is a distance measure.

Let $u_i = (1 + T(\tilde{h}_i))/\sum_{i=1}^{n}(1 + T(\tilde{h}_i))$ $\forall i = 1,2,\dots,n$, then $\sum_{i=1}^{n} u_i = 1$, and the GIVIHFPG operator given by Equation (3.1) becomes

$$\mathrm{GIVIHFPG}(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_n) = \frac{1}{\lambda}\left(\bigotimes_{i=1}^{n} (\lambda \tilde{h}_i)^{u_i}\right). \quad (3.3)$$

**Theorem 3.2** *Let* $\tilde{h}_i = \{([\mu_{\alpha_i}^-, \mu_{\alpha_i}^+], [\nu_{\alpha_i}^-, \nu_{\alpha_i}^+]) | \alpha_i \in \tilde{h}_i\} (i = 1,2,\dots,n)$ *be a collection of interval-valued intuitionistic hesitant fuzzy elements* $(IVIHFEs)$, *and a scalar* $\lambda \in (0, +\infty)$, *then, the* $GIVIHFPG$ *operator is also an* $IVIHFE$. *Moreover,*

$$\mathrm{GIVIHFPG}(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_n)$$

$$= \frac{1}{\lambda}(\bigotimes_{i=1}^{n}(\lambda\tilde{h}_i)^{(1+T(\tilde{h}_i))/\sum_{i=1}^{n}(1+T(\tilde{h}_i))})$$

$$= \bigcup_{\alpha_i\in\tilde{h}_i, i=1,2,\ldots,n}\left\{\left[1-\left(1-\prod_{i=1}^{n}(1-(1-\mu_{\alpha_i}^{-})^{\lambda})^{(1+T(\tilde{h}_i))/\sum_{i=1}^{n}(1+T(\tilde{h}_i))}\right)^{1/\lambda},\right.\right.$$

$$\left. 1-\left(1-\prod_{i=1}^{n}(1-(1-\mu_{\alpha_i}^{+})^{\lambda})^{(1+T(\tilde{h}_i))/\sum_{i=1}^{n}(1+T(\tilde{h}_i))}\right)^{1/\lambda}\right],$$

$$\left[\left(1-\prod_{i=1}^{n}(1-(\nu_{\alpha_i}^{-})^{\lambda})^{(1+T(\tilde{h}_i))/\sum_{i=1}^{n}(1+T(\tilde{h}_i))}\right)^{1/\lambda},\right.$$

$$\left.\left.\left(1-\prod_{i=1}^{n}(1-(\nu_{\alpha_i}^{+})^{\lambda})^{(1+T(\tilde{h}_i))/\sum_{i=1}^{n}(1+T(\tilde{h}_i))}\right)^{1/\lambda}\right]\right\}. \quad (3.4)$$

**Definition 3.3** Let $\tilde{h}_i = \{([\mu_{\alpha_i}^{-}, \mu_{\alpha_i}^{+}], [\nu_{\alpha_i}^{-}, \nu_{\alpha_i}^{+}]) | \alpha_i \in \tilde{h}_i\}$ $(i = 1,2,\ldots,n)$ be a collection of interval-valued intuitionistic hesitant fuzzy elements (IVIHFEs), and a scalar $\lambda \in (-\infty, +\infty), \lambda \neq 0$, then, the weighted generalized interval-valued intuitionistic hesitant fuzzy power geometric (WGIVIHFPG) operator is a mapping WGIVIHFPG: $\tilde{H}^n \to \tilde{H}$, such that

$$\text{WGIVIHFPG}(\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n) = \frac{1}{\lambda}(\bigotimes_{i=1}^{n}(\lambda\tilde{h}_i)^{w_i(1+T(\tilde{h}_i))/\sum_{i=1}^{n}w_i(1+T(\tilde{h}_i))}) \quad (3.5)$$

Futhermore,

$$\text{WGIVIHFPG}(\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n)$$
$$= \bigcup_{\alpha_i\in\tilde{h}_i, i=1,2,\ldots,n}\left\{\left[1-\left(1-\prod_{i=1}^{n}(1-(1-\mu_{\alpha_i}^{-})^{\lambda})^{\frac{w_i(1+T(\tilde{h}_i))}{\sum_{i=1}^{n}w_i(1+T(\tilde{h}_i))}}\right)^{\frac{1}{\lambda}}, 1-\left(1-\prod_{i=1}^{n}(1-(1-\mu_{\alpha_i}^{+})^{\lambda})^{w_i(1+T(\tilde{h}_i))/\sum_{i=1}^{n}w_i(1+T(\tilde{h}_i))}\right)^{1/\lambda}\right],\right.$$
$$\left.\left[\left(1-\prod_{i=1}^{n}(1-(\nu_{\alpha_i}^{-})^{\lambda})^{\frac{w_i(1+T(\tilde{h}_i))}{\sum_{i=1}^{n}w_i(1+T(\tilde{h}_i))}}\right)^{\frac{1}{\lambda}}, \left(1-\prod_{i=1}^{n}(1-(\nu_{\alpha_i}^{+})^{\lambda})^{w_i(1+T(\tilde{h}_i))/\sum_{i=1}^{n}w_i(1+T(\tilde{h}_i))}\right)^{1/\lambda}\right]\right\}$$

where $\widetilde{H}$ is the set of all IVIHFEs, $T(\tilde{h}_i) = \sum_{\substack{j=1 \\ j\neq i}}^{n} w_j Supp(\tilde{h}_i, \tilde{h}_j)$, and $w = (w_1, w_2 \dots, w_n)$ is a weighting vector with the conditions $w_i \in [0,1]$forall $i = 1,2,\dots,n$ and $\sum_{i=1}^{n} w_i = 1$.

**Definition 3.4** Let$\tilde{h}_i = \{([\mu_{\alpha_i}^-, \mu_{\alpha_i}^+], [\nu_{\alpha_i}^-, \nu_{\alpha_i}^+]) | \alpha_i \in \tilde{h}_i\}$ $(i = 1,2,\dots,n)$be a collection of interval-valued intuitionistic hesitant fuzzy elements (IVIHFEs), and a scalar$\lambda \in (-\infty, +\infty), \lambda \neq 0$, then, the generalized interval-valued intuitionistic hesitant fuzzy power ordered weighted geometric (GIVIHFPOWG) operator is a mapping GIVIHFPOWG: $\widetilde{H}^n \to \widetilde{H}$, such that

$$\mathrm{GIVIHFPOWG}(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_n) = \frac{1}{\lambda}(\bigotimes_{i=1}^{n} (\lambda \tilde{h}_{\mathrm{index}(i)})^{u_i}) \tag{3.6}$$

Furthermore,

$$\begin{aligned} &\mathrm{GIVIHFPOWG}(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_n) \\ &= \bigcup_{\alpha_i \in \tilde{h}_i, i=1,2,\dots,n} \left\{ \left[ 1 - \left(1 - \prod_{i=1}^{n} (1 - (1 - \mu_{\tilde{h}_{\mathrm{index}(i)}}^-)^{\lambda})^{u_i}\right)^{1/\lambda}, \right.\right. \\ &\left. 1 - \left(1 - \prod_{i=1}^{n} (1 - (1 - \mu_{\tilde{h}_{\mathrm{index}(i)}}^+)^{\lambda})^{u_i}\right)^{1/\lambda} \right], \\ &\left.\left[ \left(1 - \prod_{i=1}^{n} (1 - (\nu_{\tilde{h}_{\mathrm{index}(i)}}^-)^{\lambda})^{u_i}\right)^{1/\lambda}, \left(1 - \prod_{i=1}^{n} (1 - (\nu_{\tilde{h}_{\mathrm{index}(i)}}^+)^{\lambda})^{u_i}\right)^{1/\lambda} \right] \right\} \end{aligned}$$

where

$$u_i = g(\frac{R_i}{TV}) - g(\frac{R_{i-1}}{TV}), \quad R_i = \sum_{j=1}^{i} V_{\mathrm{index}(j)}, \quad TV = \sum_{i=1}^{n} V_{\mathrm{index}(i)},$$

$$V_{\mathrm{index}(i)} = 1 + T(\tilde{h}_{\mathrm{index}(i)}), \quad T(\tilde{h}_{\mathrm{index}(i)}) = \sum_{\substack{j=1 \\ j\neq i}}^{n} Supp\left(\tilde{h}_{\mathrm{index}(i)}, \tilde{h}_{\mathrm{index}(j)}\right) \tag{3.7}$$

We define the following interval-valued intuitionistic hesitant fuzzy power geometric operators as a particular case of the above mentioned generalized interval-valued intuitionistic hesitant fuzzy power geometric operators.

**Definition 3.5** Let $\lambda = 1$, then the generalized interval-valued intuitionistic hesitant fuzzy power geometric (GIVIHFPG) operator in the Definition 3.1 reduces to the interval-valuedintuitionistic hesitant fuzzy power geometric (IVIHFPG) operator, given by

$$\text{IVIHFPG}(\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n) = (\bigotimes_{i=1}^{n} (\tilde{h}_i)^{(1+T(\tilde{h}_i))/\sum_{i=1}^{n}(1+T(\tilde{h}_i))}) \quad (3.8)$$

Furthermore, for $\lambda = 1$ from Equation (3.2), we have

$$\begin{aligned}&\text{IVIHFPG}(\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n)\\ &= \bigcup_{\alpha_i \in \tilde{h}_i, i=1,2,\ldots,n} \left\{ \left[ \prod_{i=1}^{n} (\mu_{\alpha_i}^-)^{(1+T(\tilde{h}_i))/\sum_{i=1}^{n}(1+T(\tilde{h}_i))}, \prod_{i=1}^{n} (\mu_{\alpha_i}^+)^{(1+T(\tilde{h}_i))/\sum_{i=1}^{n}(1+T(\tilde{h}_i))} \right], \right.\\ &\left. \left[ 1 - \prod_{i=1}^{n} (1-\nu_{\alpha_i}^-)^{(1+T(\tilde{h}_i))/\sum_{i=1}^{n}(1+T(\tilde{h}_i))}, 1 - \prod_{i=1}^{n} (1-\nu_{\alpha_i}^+)^{(1+T(\tilde{h}_i))/\sum_{i=1}^{n}(1+T(\tilde{h}_i))} \right] \right\}\end{aligned}$$

where $\widetilde{H}$ is the set of IVIHFEs, and

$$T(\tilde{h}_i) = \sum_{\substack{j=1\\ j\neq i}}^{n} Supp(\tilde{h}_i, \tilde{h}_j) \quad (3.9)$$

$Supp(\tilde{h}_i, \tilde{h}_j)$ denotes the support of $\tilde{h}_i$ from $\tilde{h}_j$ and satisfies the three properties mentioned in Definition 3.1.

Let $u_i = (1 + T(\tilde{h}_i))/\sum_{i=1}^{n}(1 + T(\tilde{h}_i))\ \forall i = 1,2,\ldots,n$, then $\sum_{i=1}^{n} u_i = 1$, and the IVIHFPG operator becomes

$$\text{IVIHFPG}(\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n) = \bigotimes_{i=1}^{n} (\tilde{h}_i)^{u_i}$$

$$= \bigcup_{\alpha_i \in \tilde{h}_i, i=1,2,\ldots,n} \left\{ \left[ \prod_{i=1}^{n} (\mu_{\alpha_i}^-)^{u_i}, \prod_{i=1}^{n} (\mu_{\alpha_i}^+)^{u_i} \right], \right.$$

$$\left[1-\prod_{i=1}^{n}(1-\nu_{\alpha_i}^{-})^{u_i}, 1-\prod_{i=1}^{n}(1-\nu_{\alpha_i}^{+})^{u_i}\right]\Bigg\}. \tag{3.10}$$

**Corollary 3.6** *If $Supp(\tilde{h}_i,\tilde{h}_j)=C,\ \forall i\neq j$, where $C$ is a constant, then $u_i=\frac{1}{n},\ \forall i$ and*

$$\text{IVIHFPG}(\tilde{h}_1,\tilde{h}_2,\dots,\tilde{h}_n)=\text{IVIHFGA}(\tilde{h}_1,\tilde{h}_2,\dots,\tilde{h}_n)$$

$$=\bigcup_{\alpha_i\in\tilde{h}_i,i=1,2,\dots,n}\left\{\left[\prod_{i=1}^{n}(\mu_{\alpha_i}^{-})^{1/n},\prod_{i=1}^{n}(\mu_{\alpha_i}^{+})^{1/n}\right],\right.$$

$$\left.\left[1-\prod_{i=1}^{n}(1-\nu_{\alpha_i}^{-})^{1/n},1-\prod_{i=1}^{n}(1-\nu_{\alpha_i}^{+})^{1/n}\right]\right\}. \tag{3.11}$$

Thus when all the supports are the same, the IVIHFPG reduces to the interval-valued intuitionistic hesitant fuzzy geometric averaging (IVIHFGA) operator [12].

In particular, if $\mathrm{Supp}(\tilde{h}_i,\tilde{h}_j)=0,\ \forall i\neq j$, i.e. if there is no support then $u_i=1\ \forall i$ and the IVIHFPG again reduces to the interval-valued intuitionistic hesitant fuzzy geometric averaging (IVIHFGA) operator [12].

**Definition 3.7** Let $\lambda=1$, then the weighted generalized interval-valued intuitionistic hesitant fuzzy power geometric (WGIVIHFPG) operator in Definition 3.3 reduces to the weighted interval-valued intuitionistic hesitant fuzzy power geometric (WIVIHFPG) operator, given by

$$\text{WIVIHFPG}(\tilde{h}_1,\tilde{h}_2,\dots,\tilde{h}_n)=\left(\bigotimes_{i=1}^{n}(\tilde{h}_i)^{w_i(1+T(\tilde{h}_i))/\sum_{i=1}^{n}w_i(1+T(\tilde{h}_i))}\right) \tag{3.12}$$

Futhermore, for $\lambda=1$ from eq:WGIVIHFPG2, we have

$$\text{WIVIHFPG}(\tilde{h}_1,\tilde{h}_2,\dots,\tilde{h}_n)$$
$$=\bigcup_{\alpha_i\in\tilde{h}_i,i=1,2,\dots,n}\left\{\left[\prod_{i=1}^{n}(\mu_{\alpha_i}^{-})^{w_i(1+T(\tilde{h}_i))/\sum_{i=1}^{n}w_i(1+T(\tilde{h}_i))},\prod_{i=1}^{n}(\mu_{\alpha_i}^{+})^{w_i(1+T(\tilde{h}_i))/\sum_{i=1}^{n}w_i(1+T(\tilde{h}_i))}\right],\right.$$
$$\left.\left[1-\prod_{i=1}^{n}(1-\nu_{\alpha_i}^{-})^{w_i(1+T(\tilde{h}_i))/\sum_{i=1}^{n}w_i(1+T(\tilde{h}_i))},1-\prod_{i=1}^{n}(1-\nu_{\alpha_i}^{+})^{w_i(1+T(\tilde{h}_i))/\sum_{i=1}^{n}w_i(1+T(\tilde{h}_i))}\right]\right\}$$

Where $\widetilde{H}$ is the set of all IVIHFEs, $T(\tilde{h}_i) = \sum_{\substack{j=1 \\ j\neq i}}^{n} w_j Supp(\tilde{h}_i, \tilde{h}_j)$, and $w = (w_1, w_2 \dots, w_n)$ is a weighting vector with the conditions $w_i \in [0,1]$ forall $i = 1,2,\dots,n$ and $\sum_{i=1}^{n} w_i = 1$.

**Definition 3.8** Let $\lambda = 1$, then, the generalized interval-valued intuitionistic hesitant fuzzy power ordered weighted geometric (GIVIHFPOWG) operator in Definition 3.4 reduces to the interval-valued intuitionistic hesitant fuzzy power ordered weighted geometric (IVIHFPOWG) operator, given by

$$\text{IVIHFPOWG}(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_n) = \bigotimes_{i=1}^{n} (\tilde{h}_{\text{index}(i)})^{u_i} \quad (3.13)$$

Furthermore, for $\lambda = 1$ from Equation (3.7), we have

$$\text{IVIHFPOWG}(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_n)$$

$$= \bigcup_{\alpha_i \in \tilde{h}_i, i=1,2,\dots,n} \left\{ \left[ \prod_{i=1}^{n} (\mu^{-}_{\tilde{h}_{\text{index}(i)}})^{u_i}, \prod_{i=1}^{n} (\mu^{+}_{\tilde{h}_{\text{index}(i)}})^{u_i} \right], \right.$$

$$\left. \left[ 1 - \prod_{i=1}^{n} (1 - \nu^{-}_{\tilde{h}_{\text{index}(i)}})^{u_i}, \; 1 - \prod_{i=1}^{n} (1 - \nu^{+}_{\tilde{h}_{\text{index}(i)}})^{u_i} \right] \right\} \quad (3.14)$$

where

$$u_i = g(\frac{R_i}{TV}) - g(\frac{R_{i-1}}{TV}), \; R_i = \sum_{j=1}^{i} V_{\text{index}(j)}, \quad TV = \sum_{i=1}^{n} V_{\text{index}(i)},$$

$$V_{\text{index}(i)} = 1 + T(\tilde{h}_{\text{index}(i)}), \quad T(\tilde{h}_{\text{index}(i)}) = \sum_{\substack{j=1 \\ j\neq i}}^{n} Supp(\tilde{h}_{\text{index}(i)}, \tilde{h}_{\text{index}(j)}). \quad (3.15)$$

**Corollary 3.9** If $\text{Supp}(\tilde{h}_i, \tilde{h}_j) = C, \; \forall i \neq j$ and $\lambda = 1$, where C is a constant, then $u_i = \frac{1}{n}$ and the GIVIHFPG reduces to interval-valued intuitionistic hesitant fuzzy geometric averaging (IVIHFGA) operator.

### 3.1 Properties of Generalized Interval-Valued Intuitionistic Hesitant Fuzzy Geometric Power Geometric Operators

We have the following properties of the GIVIHFPG operators.

**Theorem 3.10 (Idempotency)** *Let $\tilde{h}_i$ $(i = 1,2,\dots,n)$ be a collection of interval-valued intuitionistic hesitant fuzzy elements (IVIHFEs). If $\tilde{h}_i = \tilde{h}$ for all $i$, then*

$$\mathrm{GIVIHFPG}(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_n) = \tilde{h}. \tag{3.16}$$

**Theorem 3.11 (Boundedness)** *Let $\tilde{h}_i$ $(i = 1,2,\dots,n)$ be a collection of interval-valued intuitionistic hesitant fuzzy elements (IVIHFEs), then*

$$min(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_n) \preccurlyeq GIVIHFPG(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_n) \preccurlyeq max(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_n). \tag{3.17}$$

***Theorem 3.12 (Commutativity)*** *Let $\tilde{h}_i$ $(i = 1,2,\dots,n)$ be a collection of interval-valued intuitionistic hesitant fuzzy elements (IVIHFEs). If $(\tilde{h}'_1, \tilde{h}'_2, \dots, \tilde{h}'_n)$ is a permutation of $(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_n)$, then*

$$\mathrm{GIVIHFPG}(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_n) = \mathrm{GIVIHFPG}(\tilde{h}'_1, \tilde{h}'_2, \dots, \tilde{h}'_n). \tag{3.18}$$

## 4 A MCDM method based on the GIVIHFPG operator

In this section, we propose a MCDM model based on the generalized interval-valued intuitionistic hesitant fuzzy power geometric (GIVIHFPG) operator. We assume that the evaluation information of the alternatives are given by IVIHFEs. Let $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_m\}$ be a set of alternatives and $\widetilde{H}^{(s)} = (\tilde{h}_{ij}^{(s)})_{m\times n}$ an interval-valued intuitionistic hesitant fuzzy decision matrix, where each $\tilde{h}_{ij}^{(s)}(s = 1,2,\dots,l)$ is an IVIHFE, and is given by

$$\tilde{h}_{ij}^{(s)} = \left\{\left([\mu^-_{\gamma_{ij}^{(s)}}, \mu^+_{\gamma_{ij}^{(s)}}], [\nu^-_{\gamma_{ij}^{(s)}}, \nu^+_{\gamma_{ij}^{(s)}}]\right) | \gamma_{ij}^{(s)} \in \tilde{h}_{ij}^{(s)}\right\}, (i = 1,2,\dots,m; j = 1,2,\dots,n; s = 1,2,\dots,l).$$

The process of aggregation in general involves two types of criteria, namely, one type is the benefit-type criteria, i.e., the bigger the preference values the better, and another type is the cost-type criteria, i.e., the smaller the preference values the better. The preference values of cost-type criteria can be transformed into the preference values of the benefit-type criteria. Then, the interval-valued intuitionistic hesitant fuzzy decision matrix $\widetilde{H}^{(s)} = (\tilde{h}_{ij}^{(s)})_{m\times n}$ can be transformed into the normalized interval-valued intuitionistic hesitant fuzzy decision matrix $\tilde{R} = (\tilde{r}_{ij}^{(s)})_{m\times n}$, using the method given by Xu and Hu [14], in which

$$\tilde{r}_{ij}^{(s)} = \begin{cases} \tilde{h}_{ij}^{(s)}, & \text{for benefit criteria } C_j \\ (\tilde{h}_{ij}^{(s)})^c, & \text{for cost criteria } C_j \end{cases}, (4.1)$$

where, $(\tilde{h}_{ij}^{(s)})^c (i = 1,2,\dots,m; j = 1,2,\dots,n; s = 1,2,\dots,l)$ is the complement of $\tilde{h}_{ij}^{(s)}$ such that,

$$(\tilde{h}_{ij}^{(s)})^c = \left\{\left([\nu^-_{\gamma_{ij}^{(s)}}, \nu^+_{\gamma_{ij}^{(s)}}], [\mu^-_{\gamma_{ij}^{(s)}}, \mu^+_{\gamma_{ij}^{(s)}}]\right) | \gamma_{ij}^{(s)} \in \tilde{h}_{ij}^{(s)}\right\} (4.2)$$

A MCDM model based on the proposed GIVIHFPG operator is developed as follows.

***Step 1.*** Obtain the normalized interval-valued intuitionistic hesitant fuzzy decision matrix

$\tilde{R} = (\tilde{r}_{ij}^{(s)})_{m\times n}$, using Equation set (4.1), where,

$$\tilde{r}_{ij}^{(s)} = \left\{\left([\mu^-_{\alpha_{ij}^{(s)}}, \mu^+_{\alpha_{ij}^{(s)}}], [\nu^-_{\alpha_{ij}^{(s)}}, \nu^+_{\alpha_{ij}^{(s)}}]\right) | \alpha_{ij}^{(s)} \in \tilde{h}_{ij}^{(s)}\right\},$$

$(i = 1,2,\dots,m)$; $(j = 1,2,\dots,n; s = 1,2,\dots,l)$.

***Step 2.*** Calculate the supports, $Supp\left(\tilde{r}_{ij}^{(s_1)}, \tilde{r}_{ij}^{(s_2)}\right) = 1 - d\left(\tilde{r}_{ij}^{(s_1)}, \tilde{r}_{ij}^{(s_2)}\right)$(4.3),

and $d(\tilde{r}_{ij}^{(s_1)}, \tilde{r}_{ij}^{(s_2)})$ denotes the normalized Hamming distance [23] between between any two IVIHFEs.

***Step 3.*** Calculate the weights associated with the IVIHFE $\tilde{r}_{ij}^{(s_d)} (s_d = 1,2,\dots,l; d = 1,2,3)$ using

$$\lambda_{ij}^{(s_d)} = \frac{l(1+T(\tilde{r}_{ij}^{(s_d)}))}{\sum_{s_{d^*}=1}^{l}(1+T(\tilde{r}_{ij}^{(s_{d^*})}))}, \quad (4.4), (s_d, s_{d^*} = 1,2,\dots,l; d, d^* = 1,2,3; d \neq d^*)$$

where, $T(\tilde{r}_{ij}^{(s_d)}) = \sum_{\substack{s_{d'}=1 \\ d \neq d'}}^{l} Supp(\tilde{r}_{ij}^{(s_d)}, \tilde{r}_{ij}^{(s_{d'})})$ ( 4.5),

and $d, d' = 1,2,3; d \neq d'$, and $s_d, s_{d'} = 1,2,\dots,l$. Use Equations (4.4) and (4.5) to calculate the values of $\lambda_{ij}^{s_1}, \lambda_{ij}^{s_2}, \lambda_{ij}^{s_3}; T(\tilde{r}_{ij}^{s_1}), T(\tilde{r}_{ij}^{s_2})$, and $T(\tilde{r}_{ij}^{s_3})(s_1, s_2, s_3 = 1,2,\dots,l; s_1 \neq s_2 \neq s_3)$

***Step 4.***Utilizing the GIVIHFPG operator given by Equation (3.4), in order to aggregate each of the interval-valued intuitionistic hesitant fuzzy decision matrix $\tilde{R}^{(s)} = (\tilde{r}_{ij}^{(s)})_{m\times n}(s = 1,2,\dots,l)$ into the aggregated interval-valued intuitionistic hesitant fuzzy decision matrix $\tilde{R} = (\tilde{r}_{ij})_{m\times n}$.

***Step 5.*** Calculate the supports of $\tilde{r}_{ij}(i = 1,2,\dots,m; j = 1,2,\dots,n)$ from $\tilde{r}_{it}$,

$(i = 1,2,\dots,m; t = 1,2,\dots,n)$, using

$$Supp(\tilde{r}_{ij}, \tilde{r}_{it}) = 1 - d(\tilde{r}_{ij}, \tilde{r}_{it}), \quad (4.6)$$

where, $d(\tilde{r}_{ij}, \tilde{r}_{it})(i = 1,2,\dots,m; j,t = 1,2,\dots,n; j \neq t)$ denotes the Hamming distance [23] between two IVIHFEs.

***Step 6***. Calculate the weights associated with the IVIHFEs $\tilde{r}_{ij}(i = 1,2,\dots,m; j = 1,2,\dots,n)$, using

$$\lambda_{ij} = \frac{l\left(1+T(\tilde{r}_{ij})\right)}{\sum_{s=1}^{l}\left(1+T(\tilde{r}_{is})\right)}, \quad (j, s = 1,2,\dots,l; j \neq s) (4.7)$$

$$T(\tilde{r}_{ij}) = \sum_{\substack{t=1 \\ t\neq j}}^{l} Supp(\tilde{r}_{ij}, \tilde{r}_{it}) (4.8)$$

***Step 7.*** Utilize the GIVIHFPG operator to aggregate all the IVIHFEs $\tilde{r}_{ij}(i = 1,2,\dots,m; j = 1,2,\dots,n)$ into the aggregated IVIHFEs $\tilde{r}_i(i = 1,2,\dots,m)$, similar to ***Step 4.***

***Step 8.*** Rank the aggregated IVIHFEs $\tilde{r}_i(i = 1,2,\dots,m)$, in descending order depending on the Score and the Accuracy values obtained in ***Step 7.***

***Step 9***. Calculate the score and accuracy of the IVIHFEs $\tilde{r}_i(i = 1,2,\dots,m)$, by using Definition 2.7.

***Step 10.*** Rank all the alternatives $\mathcal{A}_i(i = 1,2,\dots,m)$ according to the ranking of $\tilde{r}_i(i = 1,2,\dots,m)$ in ***Step 8.***

***Step 11.*** Select the best one(s) among the alternatives.

## 5 Illustrative Example

In this section, an illustrative example is presented to demonstrate the proposed decision making method under the interval-valued intuitionistic hesitant fuzzy environment.

Suppose a company wants to recruit a sales executive based on four available alternatives $\mathcal{A}_i(i = 1,2,3,4)$. A committee of three experts $d_s(s = 1,2,3)$ acts as decision makers (DMs) to provide their preference to the alternatives $\mathcal{A}_i(i = 1,2,3,4)$, under the following criteria, (i) $C_1$: Marketing Skills (ii) $C_2$: Technical Skills (iii) $C_3$: Communication Skills. Assume that the four alternatives $\mathcal{A}_i(i = 1,2,3,4)$, which are to be evaluated by three decision makers $d_s(s = 1,2,3)$under the criteria$C_j(j = 1,2,3)$ are expressed by the three interval-valued intuitionistic hesitant fuzzy decision matrix $\widetilde{H}^{(s)} = (\tilde{h}_{ij}^{(s)})_{4\times3}(s = 1,2,3)$ as shown in Tables 1 to 3.

Table 1: The IVIHF decision matrix $\widetilde{H}^{(1)}$ due to $d_1$

| | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|
| $\mathcal{A}_1$ | {([0.3,0.4], [0.2,0.3])} | {([0.2,0.3], [0.1,0.2])} | {([0.4,0.5], [0.3,0.4]), ([0.3,0.4], [0.5,0.6])} |
| $\mathcal{A}_2$ | {([0.1,0.3], [0.2,0.3]), ([0.2,0.3], [0.3,0.4])} | {([0.4,0.5], [0.3,0.4])} | {([0.5,0.6], [0.2,0.3]), ([0.6,0.7], [0.4,0.5])} |
| $\mathcal{A}_3$ | {([0.4,0.5], [0.5,0.6]), ([0.5,0.6], [0.4,0.5])} | {([0.6,0.7], [0.2,0.3]), ([0.7,0.8], [0.3,0.4])} | {([0.6,0.7], [0.5,0.6])} |
| $\mathcal{A}_4$ | {([0.2,0.4], [0.4,0.6])} | {([0.5,0.6], [0.3,0.5])} | {([0.5,0.6], [0.4,0.6]), ([0.4,0.5], [0.2,0.3])} |

Table 2: The IVIHF decision matrix $\widetilde{H}^{(2)}$ due to $d_2$

| | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|
| $\mathcal{A}_1$ | {([0.1,0.2], [0.2,0.3])} | {([0.2,0.3], [0.4,0.5]), ([0.2,0.4], [0.5,0.6]) | {([0.3,0.4], [0.3,0.5])} |
| $\mathcal{A}_2$ | {([0.4,0.5], [0.1,0.2]), ([0.6,0.7], [0.2,0.3])} | {([0.4,0.5], [0.2,0.3]), ([0.5,0.6], [0.4,0.5])} | {([0.3,0.5], [0.2,0.3])} |
| $\mathcal{A}_3$ | {([0.4,0.5],[0.3,0,4]), ([0.3,0.4], [0.2,0.3])} | {([0.6,0.7], [0.5,0.6])} | {([0.4,0.6], [0.3,0.4]), ([0.6,0.7], [0.2,0.3])} |
| $\mathcal{A}_4$ | {([0.6,0.7], [0.7,0.8])} | {([0.6,0.7], [0.5,0.7])} | {([0.3,0.4], [0.4,0.5])} |

Table 3: The IVIHF decision matrix $\widetilde{H}^{(3)}$ due to $d_3$

| | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|
| $\mathcal{A}_1$ | {([0.7,0.8], [0.5,0.6]), ([0.5,0.6], [0.3,0.4])} | {([0.4,0.6], [0.4,0.5])} | {([0.5,0.6], [0.7,0.8]), ([0.6,0.7], [0.3,0.4])} |
| $\mathcal{A}_2$ | {([0.7,0.8], [0.5,0.7])} | {([0.4,0.5], [0.5,0.6]), ([0.6,0.7], [0.6,0.8])} | {([0.2,0.4], [0.3,0.5]), ([0.5,0.6], [0.6,0.8])} |
| $\mathcal{A}_3$ | {([0.7,0.8], [0.3,0.5]), ([0.5,0.7], [0.2,0.3])} | {([0.3,0.6], [0.2,0.4])} | {([0.7,0.9], [0.7,0.8]), ([0.6,0.8], [0.3,0.5])} |
| $\mathcal{A}_4$ | {([0.4,0.7], [0.7,0.8]), ([0.2,0.5], [0.3,0.7])} | {([0.7,0.8], [0.4,0.5])} | {([0.2,0.5], [0.3,0.6])} |

***Step 1.*** Clearly, $C_2$ and $C_3$ are the benefit-type critera, $C_1$ the cost-type critera. we use Equation set (4.1) to find the normalized interval-valued intuitionistic hesitant fuzzy decision matrices $\tilde{R}^{(s)}, (s = 1,2,3)$ as shown in Tables 4 to 6.

Table 4: The normalized IVIHF decision matrix $\tilde{R}^{(1)}$ due to $d_1$

| | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|
| $\mathcal{A}_1$ | {([0.2,0.3], [0.3,0.4])} | {([0.2,0.3], [0.1,0.2])} | {([0.4,0.5], [0.3,0.4]), ([0.3,0.4], [0.5,0.6])} |
| $\mathcal{A}_2$ | {([0.2,0.3], [0.1,0.3]), ([0.3,0.4], [0.2,0.3])} | {([0.4,0.5], [0.3,0.4])} | {([0.5,0.6], [0.2,0.3]), ([0.6,0.7], [0.4,0.5])} |
| $\mathcal{A}_3$ | {([0.5,0.6], [0.4,0.5]), ([0.4,0.5], [0.5,0.6])} | {([0.6,0.7], [0.2,0.3]), ([0.7,0.8], [0.3,0.4])} | {([0.6,0.7], [0.5,0.6])} |
| $\mathcal{A}_4$ | {([0.4,0.6], [0.2,0.4])} | {([0.5,0.6], [0.3,0.5])} | {([0.5,0.6], [0.4,0.6]), ([0.4,0.5], [0.2,0.3])} |

Table 5: The normalized IVIHF decision matrix $\tilde{R}^{(2)}$ due to $d_2$

| | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|
| $\mathcal{A}_1$ | {([0.2,0.3], [0.1,0.2])} | {([0.2,0.3], [0.4,0.5] ([0.2,0.4], [0.5,0.6]) | {([0.3,0.4], [0.3,0.5])} |
| $\mathcal{A}_2$ | {([0.1,0.2], [0.4,0.5]), ([0.2,0.3], [0.6,0.7])} | {([0.4,0.5], [0.2,0.3] ([0.5,0.6], [0.4,0.5]) | {([0.3,0.5], [0.2,0.3])} |
| $\mathcal{A}_3$ | {([0.3,0.4], [0.4,0.5]), ([0.2,0.3], [0.3,0.4])} | {([0.6,0.7], [0.5,0.6] | {([0.4,0.6], [0.3,0.4]), ([0.6,0.7], [0.2,0.3])} |
| $\mathcal{A}_4$ | {([0.7,0.8], [0.6,0.7])} | {([0.6,0.7], [0.5,0.7] | {([0.3,0.4], [0.4,0.5])} |

Table 6: The normalized IVIHF decision matrix $\tilde{R}^{(3)}$ due to $d_3$

| | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|
| $\mathcal{A}_1$ | $\{([0.5,0.6],[0.7,0.8]),$ $([0.3,0.4],[0.5,0.6])\}$ | $\{([0.4,0.6],[0.4,0.5]$ | $\{([0.5,0.6],[0.7,0.8]),$ $([0.6,0.7],[0.3,0.4])\}$ |
| $\mathcal{A}_2$ | $\{([0.5,0.7],[0.7,0.8])\}$ | $\{([0.4,0.5],[0.5,0.6]$ $([0.6,0.7],[0.6,0.8])$ | $\{([0.2,0.4],[0.3,0.5]),$ $([0.5,0.6],[0.6,0.8])\}$ |
| $\mathcal{A}_3$ | $\{([0.3,0.5],[0.7,0.8]),$ $([0.2,0.3],[0.5,0.7])\}$ | $\{([0.3,0.6],[0.2,0.4]$ | $\{([0.7,0.9],[0.7,0.8]),$ $([0.6,0.8],[0.3,0.5])\}$ |
| $\mathcal{A}_4$ | $\{([0.7,0.8],[0.4,0.7]),$ $([0.3,0.7],[0.2,0.5])\}$ | $\{([0.7,0.8],[0.4,0.5]$ | $\{([0.2,0.5],[0.3,0.6])\}$ |

***Step 2.*** Using Equation (4.3) we calculate the supports $Supp(\tilde{r}_{ij}^{(s_1)}, \tilde{r}_{ij}^{(s_2)})$, as,

$$Supp\left(\tilde{r}_{ij}^{(1)}, \tilde{r}_{ij}^{(2)}\right) = \begin{bmatrix} 0.9500 & 0.9063 & 0.9625 \\ 0.7875 & 0.9625 & 0.9250 \\ 0.8500 & 0.9250 & 0.9188 \\ 0.8500 & 0.9250 & 0.9313 \end{bmatrix},$$

$$Supp(\tilde{r}_{ij}^{(1)}, \tilde{r}_{ij}^{(3)}) = \begin{bmatrix} 0.8750 & 0.8625 & 0.9000 \\ 0.7938 & 0.9063 & 0.8000 \\ 0.8250 & 0.9250 & 0.9313 \\ 0.9188 & 0.9375 & 0.9313 \end{bmatrix},$$

$$Supp(\tilde{r}_{ij}^{(2)}, \tilde{r}_{ij}^{(3)}) = \begin{bmatrix} 0.8250 & 0.9313 & 0.8875 \\ 0.8500 & 0.8125 & 0.8938 \\ 0.8250 & 0.8875 & 0.7750 \\ 0.9188 & 0.9375 & 0.9500 \end{bmatrix}.$$

Also, $Supp(\tilde{r}_{ij}^{(s_1)}, \tilde{r}_{ij}^{(s_2)}) = Supp(\tilde{r}_{ij}^{(s_2)}, \tilde{r}_{ij}^{(s_1)}), (s_1, s_2 = 1,2,3; s_1 \neq s_2)$.

***Step 3.*** The weights associated with the interval-valued intuitionistic hesitant fuzzy element (IVIHFE) $\tilde{r}_{ij}^{(s_d)}(s_d = 1,2,3; d = 1,2,3)$, are calculated using the Equation (4.4) and Equation (4.5) as follows,

$$\lambda_{ij}^{(1)} = \begin{bmatrix} 1.0211 & 0.9888 & 1.0103 \\ 0.9849 & 1.0292 & 0.9924 \\ 1.0031 & 1.0088 & 1.0364 \\ 0.9918 & 0.9985 & 0.9957 \end{bmatrix},$$

$$\lambda_{ij}^{(2)} = \begin{bmatrix} 1.003 & 1.0134 & 1.006 \\ 1.014 & 0.9955 & 1.0266 \\ 1.0141 & 0.9956 & 0.9795 \\ 0.9918 & 0.9896 & 1.0014 \end{bmatrix},$$

$$\lambda_{ij}^{(3)} = \begin{bmatrix} 0.9759 & 0.9978 & 0.9838 \\ 1.0088 & 0.9753 & 0.9810 \\ 0.9938 & 0.9956 & 0.9841 \\ 1.0164 & 1.0029 & 1.0022 \end{bmatrix}.$$

***Step 4.*** For sake of simplicity, we consider $\lambda = 1$ in Equation (3.4), and all the individual normalized interval-valued intuitionistic hesitant fuzzy decision matrices $\tilde{R}^{(s)} = (\tilde{r}_{ij}^{(s)})_{4\times 3}, (s = 1,2,3)$ are aggregated into the collective interval-valued intuitionistic hesitant fuzzy decision matrix $\tilde{R} = (\tilde{r}_{ij})_{4\times 3}$ is shown in Table 7.

Table 7: The collective IVIHF decision matrix $\tilde{R} = (\tilde{r}_{ij})_{4\times 3}$

| | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|
| $\mathcal{A}_1$ | {([0.1562,0.2212], [0.6422,0.6499]), ([0.1423,0.2123], [0.5152,0.5121])} | {([0.0613,0.1124], [0.5483,0.7210]), ([0.1230,0.0753], [0.5233,0.4534])} | {([0.3010,0.2442], [0.5041,0.7897]), ([0.2120,0.2568], [0.6293,0.6152]), ([0.1328,0.2418], [0.7377,0.7995])} |
| $\mathcal{A}_2$ | {([0.1240,0.1782], [0.6531,0.7546]), ([0.1235,0.2123], [0.6432,0.7852]), ([0.1423,0.2521], [0.6542,0.7574])} | {([0.2616,0.3350], [0.6216,0.6335]), ([0.2541,0.3421], [0.6431,0.7321]), ([0.2723,0.1378], [0.6742,0.7321]), ([0.2871,0.3587], [0.7540,0.7764])} | {([0.2161,0.2422], [0.4722,0.5543]), ([0.2328,0.3365], [0.6422,0.7133]), ([0.2148,0.2818], [0.5435,0.6245]), ([0.2524,0.3131], [0.6233,0.7422])} |
| $\mathcal{A}_3$ | {([0.1142,0.2461], [0.7452,0.8325]), ([0.1264,0.2215], [0.7354,0.8253]), | {([0.2664,0.3852], [0.5836,0.6654]), ([0.2895,0.4215], [0.5765,0.6424])} | {([0.2624,0.4232], [0.7524,0.8423]), ([0.2816,0.4239], [0.6212,0.7861]), |

| | | | |
|---|---|---|---|
| | ([0.1023,0.2127], [0.7542,0.8423]),<br>([0.0982,0.1425], [0.7143,0.8123]),<br>([0.1232,0.2253], [0.7452,0.8425]),<br>([0.0855,0.1852], [0.6823,0.8245])} | | ([0.3624,0.4321], [0.5812,0.6131])} |
| $\mathcal{A}_4$ | {([0.2875,0.4324], [0.6958,0.7623]),<br>([0.2423,0.4241], [0.6523,0.7545])} | {([0.3343,0.4526], [0.6753,0.7597])} | {([0.1524,0.2345], [0.5647,0.7852]),<br>([0.1254,0.1786], [0.4421,0.7435])} |

***Step 5.*** Using Equation (4.6) to (4.8), the weights associated with the IVIHFEs $\tilde{r}_{ij}(i = 1,2,3,4; j = 1,2,3)$ are calculated as follows, $(\lambda_{ij})_{4\times 3} = \begin{bmatrix} 0.9620 & 1.0321 & 0.9752 \\ 0.9734 & 1.0235 & 1.0124 \\ 1.0321 & 1.0441 & 0.9856 \\ 1.0560 & 1.0224 & 0.9463 \end{bmatrix}$(5.1)

***Step 6***. Using Equation (3.4), the IVIHFEs $\tilde{r}_{ij}$ are aggregated into the collective IVIHFEs $\tilde{r}_i(i = 1,2,3,4)$. Due to large size, the $\tilde{r}_i$'s, are not displayed here.

***Step*7**. Using Definition 2.7, the scores of the collective IVIHFEs $\tilde{r}_i(i = 1,2,3,4)$ are calculated as follows, $S(\tilde{r}_1) = 0.3923$, $S(\tilde{r}_2) = 0.3536$, $S(\tilde{r}_3) = 0.3613$, $S(\tilde{r}_4) = 0.3427$.Clearly, $S(\tilde{r}_1) > S(\tilde{r}_3) > S(\tilde{r}_2) > S(\tilde{r}_4)$. Therefore, the ranking order of the collective IVIHFEs $\tilde{r}_i(i = 1,2,3,4)$ is $\tilde{r}_1 \succ \tilde{r}_3 \succ \tilde{r}_2 \succ \tilde{r}_4$.

***Step 8.*** Since $\tilde{r}_1 \succ \tilde{r}_3 \succ \tilde{r}_2 \succ \tilde{r}_4$ the preference order of alternatives $\mathcal{A}_i(i = 1,2,3,4)$ is given by $\mathcal{A}_1 \succ \mathcal{A}_3 \succ \mathcal{A}_2 \succ \mathcal{A}_4$. Hence, $\mathcal{A}_1$ is the best alternative.

## 6 Conclusion

In this paper, we have proposed several generalized interval-valued intuitionistic hesitant fuzzy power geometric operators and some of their special cases and properties are discussed. A method for multicriteria decision making with the proposed operator is presented. Furthermore, an illustrative example is provided to demonstrate the proposed MCDM method. Alternative extensions of these operators to incorporate more realistic frameworks are part of our future research.

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**References**

[1] Lotfi, A. Zadeh. (1965) Fuzzy sets. *Information and Control*, 8(3), 338-353.

[2] Atanassov, K. T. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20(1), 87-96.

[3] Atanassov, K. T. (1989). More on intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 33(1), 37-45.

[4] Atanassov, K., & Gargov, G. (1989). Interval valued intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 31(3), 343-349.

[5] Xu, Z., & Yager, R. R. (2006). Some geometric aggregation operators based on intuitionistic fuzzy sets. *International Journal of General Systems*, 35(4), 417-433.

[6] Zeshui, X. (2007). Methods for aggregating interval-valued intuitionistic fuzzy information and their application to decision making. *Control and Decision*, 22(2), 215-219.

[7] Xu, Z. S. (2007). Models for multiple attribute decision making with intuitionistic fuzzy information. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 15(3), 285-297.

[8] Torra, V. and Narukawa, Y., (2009). On hesitant fuzzy sets and decision. *In Fuzzy systems, FUZZ-IEEE 2009. IEEE international conference on*. (pp. 1378-1382).

[9] Torra, V. (2010). Hesitant fuzzy sets. *International Journal of Intelligent Systems*, 25(6), 529-539.

[10] Xia, M., & Xu, Z. (2011). Hesitant fuzzy information aggregation in decision making. *International Journal of Approximate Reasoning*, 52(3), 395-407.

[11] Chen, N., Xu, Z., & Xia, M. (2013). Interval-valued hesitant preference relations and their applications to group decision making. *Knowledge-Based Systems*, 37, 528-540.

[12] Zhang, Z. (2013). Interval-valued intuitionistic hesitant fuzzy aggregation operators and their application in group decision-making. *Journal of Applied Mathematics*, 1-33.

[13] Xu, Z. (2007). Methods for aggregating interval-valued intuitionistic fuzzy information and their application to decision making. *Control and Decision*, 22(2), 215-219.

[14] Xu, Z., & Hu, H. (2010). Projection models for intuitionistic fuzzy multiple attribute decision making. *International Journal of Information Technology & Decision Making*, 9(2), 267-280.

[15] Xu, Z., & Chen, J. (2007). On geometric aggregation over interval-valued intuitionistic fuzzy information. *In Fuzzy Systems and Knowledge Discovery, FSKD 2007. Fourth International Conference*, 2, 466-471.

[16] Yager, R. R. (2001). The power average operator. *IEEE Transactions on Systems, Man, and Cybernetics-Part A: Systems and Humans*, 31(6), 724-731.

[17] Xu, Z., & Yager, R. R. (2010). Power-geometric operators and their use in group decision making. *IEEE Transactions on Fuzzy Systems*, 18(1), 94-105.

[18] Xu, Z. (2011). Approaches to multiple attribute group decision making based on intuitionistic fuzzy power aggregation operators. *Knowledge-Based Systems*, 24(6), 749-760.

[19] Zhou, L., Chen, H., & Liu, J. (2012). Generalized power aggregation operators and their applications in group decision making. *Computers & Industrial Engineering*, 62(4), 989-999.

[20] Zhang, Z. (2013). Generalized Atanassov's intuitionistic fuzzy power geometric operators and their application to multiple attribute group decision making. *Information Fusion*, 14(4), 460-486.

[21] He, Y. D., Chen, H., Zhou, L., Liu, J., & Tao, Z. (2013). Generalized interval-valued Atanassov's intuitionistic fuzzy power operators and their application to group decision making. *International Journal of Fuzzy Systems*, 15(4), 401-411.

[22] Zhang, Z. (2013). Hesitant fuzzy power aggregation operators and their application to multiple attribute group decision making. *Information Sciences*, 234, 150-181.

[23] Joshi, D., & Kumar, S. (2016). Interval-valued intuitionistic hesitant fuzzy Choquet integral based TOPSIS method for multi-criteria group decision making. *European Journal of Operational Research*, 248(1), 183-191.