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CLASSICAL OPERATORS OF COMPOSITE HARDY TYPE

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Abstract

In this paper an endeavour has been made to investigate classical operators of composite Hardy type on $L^p(\mu)$ space. An attempt has been made to obtain the conditions for boundedness of classical operators of Hardy type. This paper is devoted to the study of some properties of the classical operators of composite Hardy type.

1. Introduction

Given two functions $f : X \to R$ and $\varphi : X \to X$, we can produce new function by composing them under certain conditions. The resulting function is denoted by $f \circ \varphi$ and the operator $f \to f \circ \varphi$, is a linear operator C_φ known as the composition operator induced by a mapping φ. Another way to produce new function out of given two functions $\theta: X \to R$ and $f: X \to R$ is to multiply them under suitable conditions and this gives rise to the operator namely the multiplication operator M_θ induced by a function θ. Let (X, Ω, μ) be a $\sigma-$finite measure space. For each $f \in L^p(\mu)$, $1 \le p < \infty$, there exists a unique $\varphi^{-1}(\Omega)$ measurable function $E(f)$ such that

$$\int gf d\mu = \int g\, E(f) d\mu$$

for every $\varphi(\Omega)$ measurable function g for which left integral exists. The function $E(f)$ is called conditional expectation of f with respect to the sub- algebra $\varphi^{-1}(\Omega)$. Also, for expectation operators we have, $E(f) o \varphi^{-1} = g$ if and only if $E(f) = g \circ \varphi$. For wider perspective of expectation operator, one can refer to Parthasarthy [8].

Let $\varphi : X \to X$ be a non-singular measurable transformation (i.e., $\mu(E) = 0 \Rightarrow \mu\varphi^{-1}(E) = 0$). Then a composition transformation, for $1 \le p < \infty$,

$C_\varphi : L^p(\mu) \to L^p(\mu)$ is defined by

$$C_\varphi f = f \circ \varphi \qquad \text{for every} \quad f \in L^p(\mu).$$

In case, C_φ is continuous, we call it a composition operator induced by φ. The composition operators are also known as substitution operators or Koopman operator. It is easy to see that $C\varphi$ is a bounded operator if and only if

$$f_d = \frac{d\mu\varphi^{-1}}{d\mu}$$

the Radon-Nikodym derivative of the measure $\mu\varphi{-}1$ with respect to the measure μ, is essentially bounded. In 1990, Campbell [1] computed the adjoint of a composition operator on L2(μ) which is given by

$$C_\varphi^* f(x) = f_d(x) E(f o \varphi^{-1})(x)$$

Let $\theta : X \to C$ be a function such that $\theta.f \in L^p(\mu)$, for all $f \in L^p(\mu)$, then we can define a multiplication transformation $M_\theta : L^p(\mu) \to L^p(\mu)$ by

$$M_\theta \; f = \theta.f \quad \text{for every} \quad f \in L^p(\mu).$$

If M_θ is continuous, we call it a multiplication operator induced by θ. Consider the Volterra operator V acting on the Hilbert space $L^2[0, 1]$ defined by

$(Vf)(x) = \int_0^x f(y)dy$ for every $f \in L^p(\mu)$.

Thus, the Volterra operator (Hardy operator) V is an integral operator induced by the kernel k(x,y) defined as

$$k(x,y) = \begin{cases} 0; x \le 0 \\ 1; x > 1 \end{cases}$$

V^* is the adjoint of V given by

$$V^* f(x) = \int_x^1 f(t) d\mu(t).$$

Whitley [14] established the Lyubic's conjecture [7] and generalized it to Volterra composition operators on $L^p[0,1]$. The Volterra composition operator is a composition of Volterra integral operator V and a composition operator C_φ defined as

$$V_\varphi f(x) = (C_\varphi V f)(x)$$

$$V_\varphi f(x) = \int_0^{\varphi(x)} f(t) d\mu(t) \text{ for every } L^p[0,1],$$

and $\varphi: [0,1] \to [0,1]$ is a measurable function. Gupta and Komal [2] defined the Volterra composite integral operators using expectation operator as given below

$$V_\varphi f(x) = \int_0^{\varphi(x)} f(\varphi(t)) d\mu(t) \;\; = \int_0^x E(f_d \; o \; \varphi^{-1}(y) f(y) \, d\mu(y).$$

For $1 \le p < \infty$, classical Hardy operator is the map $H_\varphi: L^p[0,1] \to L^p[0,1]$ defined by

$$Hf(x) = \frac{1}{x}\int_0^x f(y)d\mu(y) \ .$$

The classical Hardy operators are also known as Cesaro operators or averaging operators. The adjoint of classical Hardy operator H^* is given by

$$H^*f(x) = \int_x^1 \frac{f(t)}{t} d\mu(t).$$

By Hardy inequality, it is well known that classical Hardy operator maps L^p into L^p. For $x \in [0,1]$, the classical Hardy operator can be written as

$$Hf(x) = \int_0^1 f(xu)du.$$

The definition of classical composite operator of Hardy type is motivated by Whitley, wherein composition operator with Volterra integral operator are composed to study Lyubic's conjecture. The classical operator of composite Hardy type is a composition of classical Hardy operator H and a composition operator C_φ defined as

$$H_\varphi f(x) = C_\varphi Hf(x) = (Hf)o\ \varphi(x)$$

$$H_\varphi f(x) = \frac{1}{x}\int_0^x f(\varphi(y))d\mu(y)$$

for every $f \in L^p[0,1]$ and $\varphi: [0,1] \to [0,1]$ is a measurable function.

Composition operators, multiplication operators and classical Hardy operators between L^p spaces have been subject matter of intensive and extensive study and they appeared prominently in operator theory, operator algebra, Orlicz spaces and dynamical systems. From the recent literature available in operator theory and functional analysis, we explore that multiplication operators and composition operators are very much intimately connected with classical Hardy operator. There exists a vast literature on the properties of these operators in different function spaces. For more recent studies, we refer to Singh and Manhas [12], Takagi [13], Halmos [3], Halmos and Sunder [4], Stepanov ([10],[11]), Kreshaw[5], Lao and Whitley[6] and Whitley [14].

Here, we recall some basic notion in operator theory. Let *H* be a Hilbert space and *B(H)* be the algebra of all bounded linear operators acting on H. We have denoted as usual by $L^p(\mu)$, the Banach space of all measurable functions $f: X \to R(or C)X$ such that

$$||f||_p = \left(\int |f(x)|^p d\mu(x)\right)^{1/p} < \infty$$

A mapping $T: X \to X$ is said to be Lipschitz continuous map if

$$||T(x-y)|| \le l\ ||x-y||$$

for some Lipschitz constant l and $x, y \in X$.

In this paper, we initiate the study of classical operators of composite Hardy type on $L^p(\mu)$ space. The characterizations for boundedness of classical operators of composite Hardy type are explored. The adjoint of classical operators of composite Hardy type is computed. The main purpose of this paper is to addresses the problem of computing the commutant of classical operators of composite Hardy type. It is proved that classical operators of composite Hardy type is one-to-one, uniformly Lipschitz continuous and Markov operator. The relation between classical operators of composite Hardy type and Volterra composite operators is also explored.

2. Boundedness and Adjoint of Classical Operators of Composite Hardy Type (COCHT)

We know that C_φ is bounded operator, therefore we have

$$||C_\varphi f||^p = \int |(f o \varphi)(x)|^p \, d\mu(x)$$

$$= \int |f(x)|^p \, d\mu\varphi^{-1}(x)$$

$$= \int |f(x)|^p \frac{d\mu\varphi^{-1}}{d\mu} d\mu$$

$$\leq \alpha \, ||f||^p,$$

where $f_d = \frac{d\mu\varphi^{-1}}{d\mu} \leq \alpha.$

Theorem 2.1: Let $H_\varphi \in B(L^1[0,1])$ and $f \in L^1[0,1]$. Then, for $y \in [0,1]$

$$||H_\varphi f||_1 \leq \alpha \, \log\frac{1}{y} \, ||f||_1$$

Proof: For $f \in L^1[0,1]$, we have

$$||H_\varphi f||_1 = \int_0^1 |H_\varphi \, f(x)| d\mu(x)$$

$$= \int_0^1 | \, \frac{1}{x}\int_0^x f\big(\varphi(y)\big) d\mu(y) \, | \, d\mu(x)$$

$$\leq \int_0^1 (\int_y^1 \frac{1}{x} d\mu(x)) \, | \, f(\varphi(y))| d\mu(y)$$

$$= \log\frac{1}{y}\int_0^1 | \, f(\varphi(y))| d\mu(y)$$

$$= \log\frac{1}{y} \, ||C_\varphi f||_1$$

Thus, we have

$$||H_\varphi f||_1 \leq \alpha \, \log\frac{1}{y} \, ||f||_1.$$

In the next result, we explore the condition for boundedness of classical operators of composite Hardy Type (COCHT) on $L^p[0,1]$.

Theorem 2.2: For $1 < p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1$, let $H_\varphi \in B(L^p[0,\infty))$. Then H_φ is is a bounded linear operator and

$$||H_\varphi f||_p \leq \alpha \, \frac{p}{1-p} \, ||f||_p$$

where $f_d = \frac{d\mu\varphi^{-1}}{d\mu} \leq \alpha$.

Proof: For $f \in L^p[0,\infty)$, we have

$$||H_\varphi f||_p^p = \int_0^\infty |H_\varphi f(x)|^p d\mu(x)$$

$$= \int_0^\infty \left| \frac{1}{x}\int_0^x f(\varphi(y)) d\mu(y) \right|^p d\mu(x)$$

$$= \frac{1}{1-p}\left[(1-p)\int_0^\infty \left| x^{-p}\left(\int_0^x f(\varphi(y))d\mu(y)\right)^p \right| d\mu(x)\right]$$

$$= \frac{1}{1-p}\left[\int_0^\infty \left| \left(\int_0^x f(\varphi(y))d\mu(y)\right)^p \right| d\mu(x^{1-p})\right]$$

$$= \frac{1}{1-p}\left[x^{1-p}\int_0^\infty \left| \left(\int_0^x f(\varphi(y))d\mu(y)\right) \right|^p \Big|_0^1 \, d\mu(x)\right] -$$

$$p\left[\int_0^\infty \left[\left(\int_0^x f(\varphi(y))d\mu(y)\right)^{p-1} f(\varphi(x))\right] x^{1-p} d\mu(x)\right]$$

$$\leq \frac{p}{1-p}\left\{\left[\int_0^x f(\varphi(y))^p d\mu(y)\right]^{1/p} \frac{1}{x}\left[\int_0^\infty |H_\varphi f(x)|^{q(p-1)} d\mu(x)\right]^{1/q}\right\}$$

$$= \frac{p}{1-p}||fo\varphi||_p \left(||H_\varphi f||_p\right)^{p/q}$$

Again, we have

$$||H_\varphi f||_p^p \leq \alpha \, \frac{p}{1-p} \, ||f||_p \left(||H_\varphi f||_p\right)^{p/q}$$

$$||H_\varphi f||_p \leq \alpha \, \frac{p}{1-p} \, ||f||_p$$

Hence, we get the required result

$$||H_\varphi f||_p^{p-\frac{p}{q}} \leq \alpha \, \frac{p}{1-p} \, ||f||_p$$

Again, for scalars a, b and $f \in L^p[0,\infty)$, we have

$$H_\varphi(af + bg)(x) = \frac{1}{x}\int_0^x (af + bg)(\varphi(y)) d\mu(y) \; .$$

$$= \frac{1}{x}\int_0^x (af(\varphi(y))d\mu(y) + \frac{1}{x}\int_0^x (bg(\varphi(y))d\mu(y) \; .$$

$$= aH_\varphi f(x) + bH_\varphi g(x).$$

In the next result, we have obtained the adjoint of COCHT H_φ .

Theorem 2.3: Let $H_\varphi \in B(L^2[0,1])$. Then the adjoint of H_φ

$$H_\varphi^* f = f_d \, E(\, H^* f)o\varphi^{-1}$$

Proof: Let $f, g \in L^2[0,1]$. Then, we have

$$\langle f, H_\varphi g\rangle = \int_0^1 f(x)\overline{H_\varphi g(x)}\, d\mu(x)$$

$$= \int_0^1 f(x)\frac{1}{x}\int_0^x \overline{go\varphi(y)} \;\; d\mu(y)d\mu(x)$$

$$= \int_0^1 \overline{go\varphi}(y)\left[\int_y^1 \frac{f(x)}{x}\, d\mu(x)\right]d\mu(y)$$

$$= \int_0^1 \overline{go\varphi}(y) \;\, (H^* f)(y)d\mu(y)$$

$$= \langle H^* f, C_\varphi \, g\rangle$$

$$= \langle \; C_\varphi^* \, H^* f, \; g\rangle$$

Therefore,

$H_\varphi^* f(x) = \; C_\varphi^* \, H^* f(x) = \; f_d \, E(H^* f o\varphi^{-1})(x)$

Hence, we have

$$H_\varphi^* f = f_d \, E(\, H^* f)o\varphi^{-1}.$$

3. Algebraic Properties of Classical Operators of Composite Hardy Type (COCHT)

Theorem 3.1: Let $H_\varphi \in B(L^2[0,1])$. Suppose A is unitary operator. If $f : B(L^2[0,1]) \rightarrow B(L^2[0,1])$ is defined as $f(H_\varphi) = A^* H_\varphi A$. Then f is an isometry.

Proof: Let $H_\varphi \in B(L^2[0,1])$ and I be an identity operator. Then, we have

$||H_\varphi|| = \; ||IH_\varphi I||$

$= \; ||AA^* H_\varphi AA^*||$

$\leq ||A|| \; ||A^* H_\varphi A||||A^*||$

$= f(H_\varphi)$.

Again, we have

$$||f(H_\varphi)|| = \left||A^* H_\varphi A\right||$$

$$\leq ||A^*|||\left|H_\varphi\right|||A|| = ||H_\varphi||$$

Hence, we have

$$||f(H_\varphi)|| = ||\, A^* H_\varphi A|| = \left||H_\varphi\right||.$$

In the next example we show that H_φ is not isometry on L^p - space in general.

Example 3.2: Let $X = R^+$ and $\varphi: R^+ \to R^+$ be a translation map defined as

$$\varphi(x) = ax, a \in R^+, 0 \neq a \neq 1.$$

Clearly, $f_d = \frac{1}{a}$. Now, we have

$$||H_\varphi f||_p^p = \int |H_\varphi\, f(x)|^p \, d\mu(x)$$

$$= \int_0^\infty \left| \, \frac{1}{x}\int_0^x fo\varphi(y) dy \, \right|^p d\mu(x)$$

$$\leq \int \frac{1}{x^p}\left[\int_0^1 |\, fo\varphi(y)\, |^p dy\right] dx$$

$$\leq \frac{1}{a}\int \frac{1}{x^p}\left[\int_0^1 |\, f(y)\, |^p dy\right] dx$$

$$= \frac{1}{a}\int \frac{1}{x^p} dx \; ||x||_p^p$$

which shows that H_φ is not an isometry.

Theorem 3.3: Let $H_\varphi \in B(L^2[0,1])$. Then H_φ is uniformly Lipschitz continuous with Lipschitz constant $||H_\varphi||$.

Proof: Let $f, g \in L^2[0,1]$, by using linearity of H_φ we have

$$||H_\varphi f - H_\varphi g|| = ||H_\varphi(f - g)||$$

$$= ||f - g|| \left[\frac{||H_\varphi(f - g)||}{||f - g||}\right]$$

$$\leq \left||f - g\right||\; \left||H_\varphi\right||.$$

Hence, H_φ is uniformly Lipschitz continuous with Lipschitz constant $||H_\varphi||$.

In the next result, we have explored the relation between composite Hankel operators H_φ and Volterra composite operators V_φ.

For $x \in [0,1]$, we define $A_x : L^2[0,1] \to L^2[0,1]$ as $A_x f(x) = xf(x)$.

Theorem 3.4: Let $H_\varphi \in B(L^2[0,1])$. Then

$$A_{x^m} V_\varphi = A_{x^{m+1}} H_\varphi$$

Proof: For $f, g \in L^2[0,1]$, we have

$A_{x^{m+1}} H_\varphi f(x) = x^{m+1}(H_\varphi f)(x)$

$= x^{m+1} \left[\frac{1}{x} \int_0^x f(\varphi(y)) d\mu(y)\right]$

$= x^m \int_0^x f(\varphi(y)) d\mu(y)$

$= x^m V_\varphi f(x)$

$= A_{x^m} V_\varphi f(x)$

Hence, the result is proved.

Theorem 3.5: Let $H_\varphi \in B(L^2[0,1])$. Suppose e is a constant function such that $e(x) = 1$ for all $x \in [0,1]$. Then $H_\varphi^n e = 1$ for each positive integer n.

Proof: For $x \in [0,1]$, we have $e(x) = 1$. e(x) = 1.

For $n = 1$, we have

$H_\varphi e(x) = \left[\frac{1}{x} \int_0^x e(\varphi(y)) d\mu(y)\right]$

$= \frac{1}{x} \int_0^x d\mu(y) = 1$

For $n = 2$, we have

$H_\varphi^2 e(x) = H_\varphi (H_\varphi e(x)) = H_\varphi 1 = H_\varphi e(x).$

By the principle of mathematical induction, the result is true for $n = k$.

Now for $n = k + 1$, we have

$H_\varphi^{k+1} e(x) = H_\varphi H_\varphi^k e(x)) = H_\varphi 1 = H_\varphi e(x) = 1.$

Hence, the result is true for each positive integer n.

Theorem 3.6: Let $H_\varphi \in B(L^2[0,1])$. Suppose φ is an injective map and M_θ is a multiplication operator induced by θ. Then H_φ commutes with M_θ if and only if $\theta = \theta o \varphi$ almost everywhere.

Proof: Given $f, g \in L^2[0,1]$, we have

$$H_\varphi M_\theta f(x) = \frac{1}{x}\int_0^x (M_\theta f)(\varphi(y))d\mu(y)$$

$$= \frac{1}{x}\int_0^x \theta(\varphi(y))f(\varphi(y))d\mu(y)$$

Again, we have

$$M_\theta H_\varphi f(x) = \theta(x)\frac{1}{x}\int_0^x f(\varphi(y))d\mu(y)$$

Now, we have

$$H_\varphi M_\theta f(x) - M_\theta H_\varphi f(x) = \frac{1}{x}\int_0^x f(\varphi(y))[\theta(\varphi(y)) - \theta(x)]d\mu(x)$$

as given φ is injective, so C_φ has dense range.

Hence, the desired result follows.

In the following example, we show that on L^p – space, classical operators of composite Hardy type H_φ does not have closed range.

Example 3.7: Let $\varphi: [0,1] \to [0,1]$ be defined as $\varphi(x) = \sqrt{x}, \forall x \in [0,1]$.

Then $\varphi^{-1}(x) = x^2$ and $f_d(x) = 2x$.

Suppose E_n is the set which contains all those $x \in [0,1]$ such that $f_n(x) \leq \frac{1}{n}$.

Then, $\mu(E_n) > 0$. Consider $f_n = \frac{\chi_{E_n}}{\sqrt[p]{\mu(E_n)}}$.

Then $||f_n||_p = 1$.

Also, we have

$$||H_\varphi f_n||_p = \left[\int_0^1 |\frac{1}{x}\int_0^x (f_n o\varphi)(y)d\mu(y)|^p\, d\mu(x)\right]^{1/p}$$

$$\leq \left[\int_0^1 \frac{1}{x^p}\left(\int_0^x |f_n(y)|^p\, f_d(y)d\mu(y)\right)d\mu(x)\right]^{1/p}$$

$$= \left[\int_0^1 \frac{1}{nx^p}\left(\int_0^x |\frac{\chi_{E_n}}{\sqrt[p]{\mu(E_n)}}|^p\, f_d(y)d\mu(y)\right)d\mu(x)\right]^{1/p}$$

$$= \frac{1}{n}\int_0^1 \frac{1}{x}\, d\mu(x)$$

$$= \frac{1}{n} \log x \ |_0^1$$

which tends to 0 as n tends to ∞. Hence, H_φ does not have closed range.

4. Conclusion:

In this paper, formulation of classical operators of composite Hardy type and its study has explored the characterizations for boundedness of classical operators of composite Hardy type and the adjoint of classical operators of composite Hardy type is computed. The problem of computing commutant of classical operators of composite Hardy type is addressed. Algebraic properties for of classical operators of composite Hardy type operators like, one-to-one mapping, uniformly Lipschitz continuous and Markov operator are derived.

The study may open new horizons to explore more properties of classical operators of composite Hardy type.

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