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AN EVIDENCE OF FEIGENBAUM UNIVERSALITY IN A CHAOTIC REGION

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Abstract

In one of his seminal papers [6], in the year 1982, M. J. Feigenbaum, one of the pioneers of Chaos theory, derived an equation (equation 19 of the paper) which established a relationship between the 'Misiurewicz points' and 'Lyapunov exponents'. This equation predicts that n-th Lyapunov exponent is equal to 2^n *times the Lyapunov exponent at a specific point* x_n *of the n times renormalized map. Those n times renormalized maps converge to the universal 'Feigenbaum fixed point' map (call it* F*). The points* x_n *converge to a specific point y (all those points are defined dynamically). Therefore, the n-th Lyapunov exponent multiplied by* 2^n *converges to the Lyapunov exponent of* F *at* y*. Moreover, this fact is independent of the family of maps, provided those are unimodal and the second derivative at the critical point is non-zero. In our present work, we have calculated the Misiurewicz points and determined the values of the Lyapunov exponent at those points numerically for the Logistic map. Further, the relationship which was predicted by Feigenbaum in [6] in between the Misiurewicz* *points and the Lyapunov exponents at those points is numerically verified.*

Keywords:reverse bifurcation, misiurewicz points, lyapunov exponent, chaos.

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1. Introduction

M.J. Feigenbaum, (December 19, 1944 – June 30, 2019) was an American mathematical physicist whose pioneering studies in chaos theory led to the discovery of the Feigenbaum universality and Feigenbaum constants. Though later on it was recognised as a universality for a certain type of systems, Feigenbaum derived his remarkable mathematical relations, working on the simple looking Logistic map. It was remarkable in the sense that though the map looks quite simple, it can show wide range of behaviours including chaotic one with variation in the control parameter of the map.

For ready reference of the readers, the Logistic map is given by$f(x) = \mu x(1-x)$, where $\mu \in [0,4]$ is the growth rate of a population$x \in [0,1]$. It has long been studied as a simple but illustrative case of nonlinear maps which can show chaotic behaviour through a series of period doubling bifurcations when the control parameter μ of the map is changed from lower to upper admissible values. Research journey on Logistic map began with R. M. May in 1975 but not yet completed [13, 31]. In recent times chaotic Logistic map is used as pseudo random bit generator [25] and in image encryption [15], wireless communication [2], Artificial Neural Networks for Code Generation [1]. A modified Logistic Map is used for Chaotic Mobile Robot’s Path Planning [26].

2. Lyapunov Exponent :

One of the salient features of chaos is the divergence of nearby trajectories. In fact, for a chaotic system, this divergence is exponential in time (for continuous system) or iteration number (for discrete system). Exponential convergence or divergence of nearby trajectories or orbits in phase space signify the qualitative dynamical behaviour of a system and Lyapunov exponents are mathematical tools to signify and measure it [32]. For one dimensional maps, negative Lyapunov exponent signify deterministic or regular behaviour and a positive Lyapunov exponent gives indication that the behaviour becomes chaotic. For higher dimensional systems, one or more positive Lyapunov exponents indicate chaotic behaviour.

The Lyapunov exponent λ of a one dimensional map $x_{n+1} = f(\mu, x_n)$ is obtained operationally by iterating the map, keeping track of the average natural logarithm of the slope. It is measured as

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \ln|f'(\mu, x_k)|$$

A plot of λ as a function of the parameter for the Logistic map $x_{n+1} = \mu x_n(1 - x_n)$ was presented for the first time by Shaw [30]. In 1981, Gieselet. al. [7] discussed the dependence of Lyapunov exponents of stable and unstable periodic orbits of the Logistic map on the control parameter, using an approximate renormalization procedure. Huberman and Rudnick [10] studied chaotic bands and established a power law for Lyapunov exponents as a function of the control parameter. In a one dimensional quadratic map, there are three types of points : points where $\lambda < 0$ (stable and superstable Periodic points), where $\lambda = 0$ (bifurcation points) and points where $\lambda > 0$ (Misiurewicz points) [18-20].

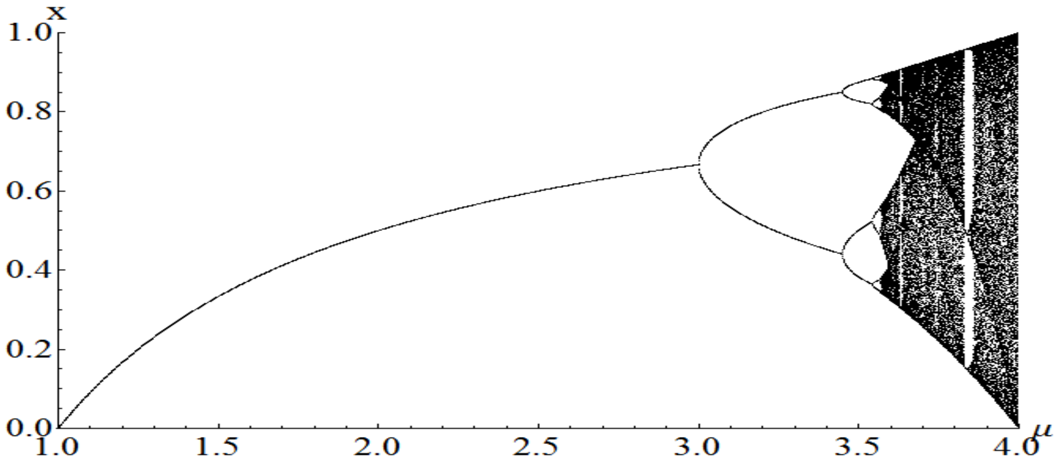

Fig. 1: Bifurcation diagram of the logistic map

There exists an ‘inverse cascade’ of band merging or band splitting (depending on whether the parameter is changed from lower to upper or upper to lower values) of chaotic bands for those systems which undergo a ‘period doubling cascade’. This phenomenon is termed as ‘reverse bifurcation’ [21-24]. Importance of this reverse bifurcation scenario is that it shows us how a model can change its behaviour from chaotic to a regular one. Kuruvilla and Nanda Kumaran [12] used the technique of reverse bifurcation to suppress chaos in coupled directly modulated semi-conductor. The Chaotic band merging or splitting points in case of reverse bifurcation points are called Misiurewicz points [14,29] and those points strictly have a positive Lyapunov exponent in case of one dimensional maps. Pastor et.al [18-24, 27], studied the phenomenon of reverse bifurcation extensively and produced a series of papers on Misiurewicz points and reverse bifurcation.

On the basis of the earlier mentioned equation (equation 19 of [6]), Feigenbaum analytically predicted about a functional relationship among the Lyapunov exponents at those points. In fact, both theoretical understanding of the phenomenon of reverse bifurcation and numerical calculations are required to locate the Misiurewicz points. Sarmah et al. [28], have discussed the technique quite elaborately. In this paper, our main concern is to numerically verify the prediction of Feigenbaum in case of the Logistic map [13]. We have chosen the map because it is the prototype of one dimensional unimodal maps which can show chaotic behaviour through a cascade of period doubling bifurcations. In this paper, we have listed the Misiurewicz points and the Lyapunov exponents at those Misiurewicz points, which were found numerically and have verified the relationship predicted by Feigenbaum analytically.

3. Reverse bifurcation and Misiurewicz points :

In Fig. 1, we have plotted the bifurcation diagram of the Logistic map. To the left of the Feigenbaum point or the accumulation point the map displays a stable periodic behavior. For the region $1 < \mu < 3$, the stable attractor is a point attractor and its value increases as the value of μ increases. At $\mu = 3$, the first period doubling bifurcation occurs. For $\mu > 3$, the attractor splits into two, showing a period-2 cycle indicated by the two branches. With increase in values of the control parameter μ, both branches again split simultaneously giving birth to period 4 cycles at the next period doubling bifurcation point. This splitting phenomena is the period doubling bifurcation. As μ increases, a cascade of further period-doublings occur, yielding period-8 cycle, period-16 cycle and so on, until at $\mu = \mu_\infty \approx 3.56994567$ ..., the map becomes chaotic and the attractor changes from a finite to an infinite set of points [5].

Investigation of many researchers unearthed that similar phenomena occurs inside the chaotic region also. But in this case it is the chaotic band splitting in the reverse order. If we observe the chaotic region of the bifurcation diagram of the Logistic map from lower to upper parameter values, then it is seen that near $\mu = 3.68$, two chaotic bands merge into one and there seems to be a convergence of the curves of the image points. This special image point is called the Misiurewicz point [9,14]. On the other hand, if we observe the same from higher parameter values to lower ones, we can say that near $\mu = 3.68$, one chaotic band splits into two chaotic bands. This phenomenon, in scientific literature, is called the reverse bifurcation or the band splitting bifurcation [16]. The same type of crossing or convergence occurs near $\mu = 3.6$, where two chaotic bands split to give four chaotic bands. After n reverse bifurcations, the attractor is partitioned in 2^n chaotic bands, which are self-affine copies of the first chaotic band [4, 17].

On careful investigation, it is seen that there are some dark 'curves' of points that run through the chaotic region of the bifurcation diagram. These dark curves are the images of the critical point $x_c = \frac{1}{2}$ of the Logistic map. In figure 2(a) we have plotted the first ten images of the critical point and in figure 2(b) these images are superimposed on the bifurcation diagram.

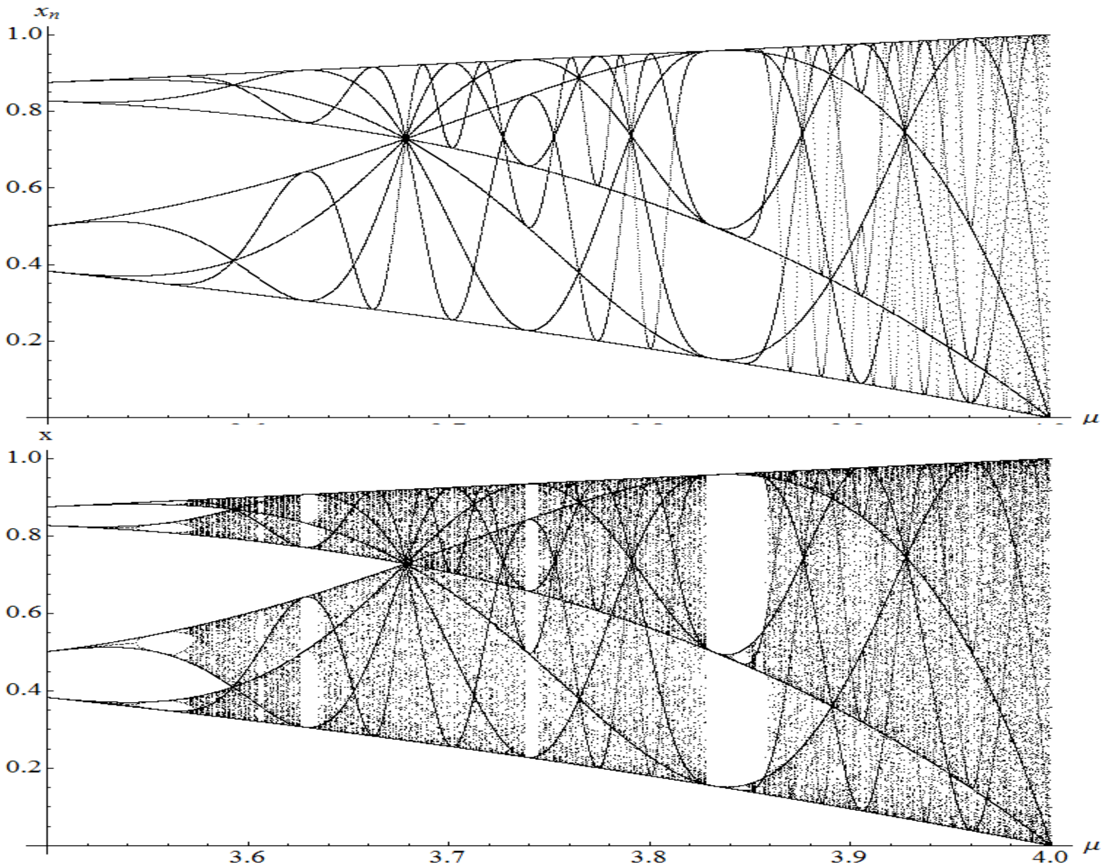

Fig. 2: (a) First ten images of the critical points in the range $3.5 \leq \mu \leq 4$. (b) The location of the first ten images of the critical point (boundaries) are plotted on the bifurcation diagram

From Fig. 2, it is observed that the odd iterations form the upper branches while the even iterations form the lower branch in the bifurcation diagram. The first and the second iterates of the critical point formed the outer boundaries of the iterative process after a transient period. The first iteration gives the upper boundary whereas second iteration gives the lower boundary. All other higher order iterates of the function will lie in between the first and second iterates of the Logistic map. In the region from μ_∞ to the first Misiurewicz point, the inner most iterates will be the third iterate and the fourth iterate of the map function at its critical point, in which $f^3(x_c)$ is the lower boundary of the upper chaotic band and $f^4(x_c)$ is the upper boundary of the lower chaotic band as shown in the figures 2(a) and 2(b) and both these iterate intersect at the first Misiurewicz point. Therefore, the first Misiurewicz point can be obtained by solving the equation

$$f^3(x_c) - f^4(x_c) = 0 \qquad (1)$$

The equation (1) finally reduces to $\mu^3 - 2\mu^2 - 4\mu - 8 = 0$ which gives the first Misiurewicz point as $3.678573510428 \ldots$. Using similar technique, which was discussed in detail in the paper Sarmah et.al. [28], we have determined the first 10 Misiurewicz points of the Logistic map which are shown below in the table I.

In between μ_∞ and the first Misiurewicz point there may be a large number of separate bands (say, approximately 2^∞ bands near μ_∞) which merge together as the parameter varies and finally becomes a single band at the above mentioned Misiurewicz point. This band merging process takes place only when an unstable fixed point hits the attractor. The band merging of order 2^n to 2^{n-1} with n =1, 2, 3… takes place only when the unstable fixed points, which were created during the bifurcation of 2^{n-1} cycle to 2^n cycle, hits the chaotic attractor or band of order 2^n [3, 8]. Below we have shown this fact in case of the first three Misiurewicz points.

Table I : The first 10 misiurewicz points of the logistic map

Number of Chaotic bands born	Misiurewicz points (m_i)
2	3.678573510428322…
4	3.592572184106978…
8	3.574804938759208…
16	3.570985940341614…

32	3.570168472496375…
64	3.569993388559133…
128	3.569955891325219…
256	3.569947860564655…
512	3.569946140622108…
1024	3.569945772263088…

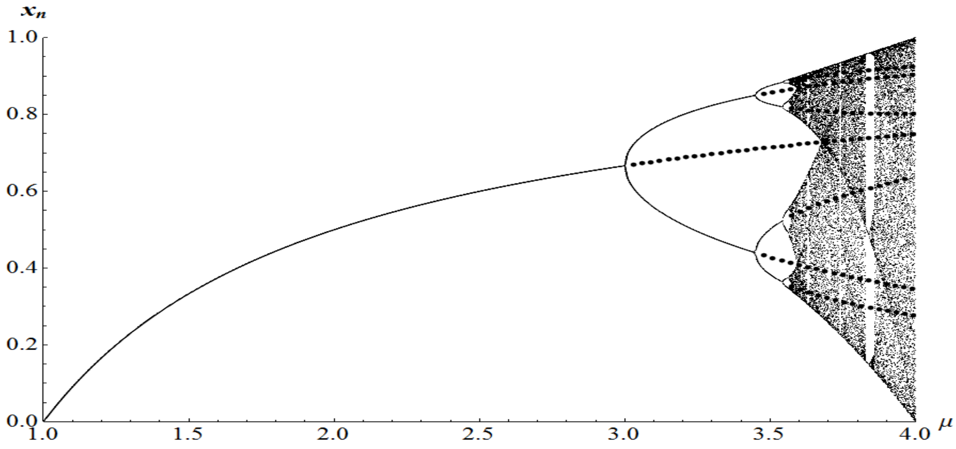

Fig. 3: Bifurcation diagram of the logistic map with unstable periodic orbits

The dotted line shows the unstable periodic orbit created at every period doubling bifurcation. It is to be noted that corresponding to every period doubling bifurcation there exists a Misiurewicz point.

To the right of the Feigenbaum point, which is also known as accumulation point, a completely different zone can be observed formed mostly by chaotic bands. At the same time, some sort of similarity can also be observed from an entirely different perspective. Towards the right extreme, for $\mu = 4$, there is only one chaotic band spanning the whole interval from 0 to 1. It is the so called 1-periodic chaotic band. When μ decreases the band narrows and at $\mu = m_1$(say) the band splits into two parts that compose the $2(= 2^1)$ chaotic bands. At $\mu = m_2$(say) these two chaotic bands again split into four parts that compose the $4(= 2^2)$ chaotic bands and so on.

Therefore, there is also a period doubling cascade of chaotic bands that finishes from the opposite side at the Feigenbaum point $\mu_\infty = m_\infty = 3.56994567 \ldots$.[19]. This region is called the chaotic region, and points m_i are called band merging points or Misiurewicz points.

4. Lyapunov exponent and Misiurewicz points:

In Fig. 4 we have plotted the Lyapunov exponents of the Logistic map for the parameter values $3 \leq \mu \leq 4$. To the right of the accumulation point the region is chaotic [5] which is clear from the plot of the Lyapunov exponent of the Logistic map.

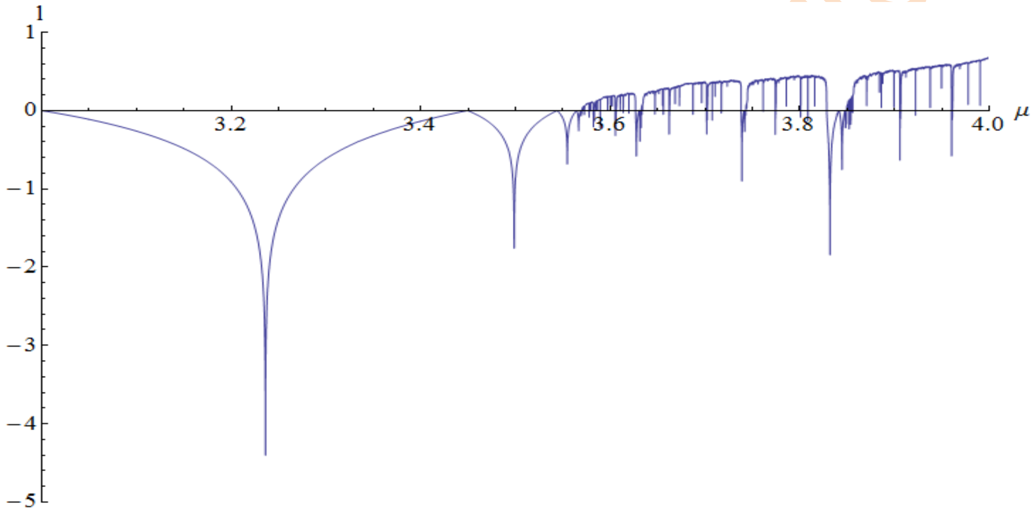

Fig 4: Lyapunov exponents of the logistic map for $3 \leq \mu \leq 4$

For this region, the values of the Lyapunov exponents are mostly positive with appearance of negative values sporadically indicating the presence of periodic windows (regular behaviour of the system) inside the chaotic region. We have calculated the Lyapunov exponents at each of the Misiurewicz points up to 12 decimal places and these are listed in the Table 2 below.

Table II: Table of misiurewicz points and lyapunov exponents

Misiurewicz points(m_i)	Misiurewicz points(m_i)
3.678573510428322…	0.342256545061 ...
3.592572184106978…	0.171751902216 ...

3.574804938759208…	0.085803955232 ...
3.570985940341614…	0.042898718972 ...
3.570168472496375…	0.021451674934 ...
3.569993388559133…	0.010731081009 ...
3.569955891325219…	0.005359334177 ...
3.569947860564655…	0.002686878311 ...
3.569946140622108…	0.001339095685 ...
3.569945772263088…	0.000668974806 ...

From Table II, it is seen that if we consider up to 5 or 6 decimal places, each subsequent Lyapunov exponent is almost half of the preceding one. Denoting the Lyapunov exponents at the Misiurewicz points by $l_0, l_1, l_2, l_3, l_4, l_5, \dots\dots, l_n, \dots\dots$ respectively, we get a functional relation of the form $l_n = \frac{l_0}{2^n}$ which was derived theoretically by M.J. Feigenbaum in [6]. The Lyapunov exponent ultimately converges to zero at the accumulation point μ_∞, where the number of chaotic bands become infinite.

5. Conclusion

In this paper, we have numerically verified a relationship between the Misiurewicz points of the Logistic map and the Lyapunov exponents at those points which was predicted by M.J Feigenbaum in one of his most famous papers [6]. Further, the phenomenon of reverse bifurcation and the idea of Misiurewicz points and the technique to find those points numerically have been discussed in brief. For detailed discussion, interested readers can go through our earlier mentioned paper [28]. For application purpose of the chaotic behaviour of the Logistic map in some fields of modern science and technology, interested readers are referred to [1, 2, 11, 15, 25, 26, 31].

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