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# DEGREE PRODUCT ADJACENCY ENERGIES OF COMPLEMENT OF REGULAR GRAPHS AND COMPLEMENT OF LINE GRAPHS OF REGULAR GRAPHS.

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**Abstract**

*In this article, we find the explicit formulas for the degree product adjacency energy of the complement graph of a $r$ regular graph and also the degree product adjacency energy of $\overline{L(G)}$. In this way one can calculate/compute the degree product adjacency energy of large family of regular graphs.*

***Keywords:*** Degree product adjacency energy, complement of a graph, line graph.

**2010 AMS classification:** 05C05, 05C50

## 1. Introduction

Graphs considered in this article are simple, connected with $n$ vertices and $m$ edges, $d_i$ is the degree of the vertex $v_i$. For undefined terminologies we refer [6].

The graph $G$ is a regular graph, where all its vertices are equal to degree $r$. The complement $\overline{G}$ of a graph $G$ also has $n$ number of vertices but two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in $G$. The line graph $L(G)$ is a graph, in this the number of vertices are equal to the number of edges of graph $G$ and any two vertices of $L(G)$ are adjacent if and only if the corresponding edges in $G$ are adjacent [6].

The adjacency matrix of a graph $G$ is a square matrix and is defined as $A(G)$ = $[a_{ij}]$, where $a_{ij}$ is [1],

$$a_{ij} = \begin{cases} 1, & if v_i \sim v_j; \\ 0, & otherwise. \end{cases} \qquad (1)$$

Where the notation $v_i \sim v_j$ stands for the vertex $v_i$ is adjacent to vertex $v_j$. The eigenvalues of the adjacency matrix of $G$ are denoted by $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$.

The energy of a graph $G$ is defined as the sum of the absolute values of the eigenvalues of adjacent matrix of graph $G$. This concept was introduced by I. Gutman [4]. This energy has been well explained in [5] and its mathematical representation is,

$$E_A(G) = \Sigma_{i=1}^{k} |\lambda_i|$$

The degree product adjacency energy $E_{DPA}(G)$ is defined as follows [7],

The $DPA(G)$ is the degree product adjacency matrix and is defined as,

$$d_{ij} = \begin{cases} d_i d_j, & if v_i \sim v_j; \\ 0, & otherwise. \end{cases}$$

The degree product adjacency matrix $DPA(G)$ is a real symmetric matrix and its eigenvalues are $\alpha_1, \alpha_2, \alpha_3 \ldots, \alpha_k$. The order of eigenvalues be $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \ldots \geq \alpha_k$. The similar way of adjacency energy, the degree product adjacency energy of a graph defined as [7],

$$E_{DPA}(G) = \Sigma_{i=1}^{k} |\alpha_i| \qquad (2)$$

[1] The spectrum of a graph $G$ is the set of numbers, which are eigenvalues of adjacency matrix $A(G)$, together with their multiplicities. Analogues to spectrum of $A(G)$, the spectrum of degree product adjacency matrix is defined as [7],

$$Spec(DPA)(G) = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_k \\ m_1 & m_2 & m_3 & \dots & m_k \end{pmatrix} (3)$$

where $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \dots \geq \alpha_k$ are the eigenvalues of $DPA(G)$ matrix and $m_1, m_2, m_3, \dots, m_k$ are multiplicities of $\alpha_1, \alpha_2, \dots, \alpha_k$ respectively. Here $m_1 + m_2 + m_3 + \dots + m_k = n$

The following theorems are used to prove the main results.

**Theorem 11.1.**[1] *Let $G$ be a $r$ regular graph with spectra of adjacency matrix as,*

$$Spec(G) = \begin{pmatrix} r & \lambda_2 & \lambda_3 & \dots & \lambda_k \\ 1 & m_2 & m_3 & \dots & m_k \end{pmatrix}$$

Then $\overline{G}$, the complement of $G$ is a $(n - r - 1)$ regular graph with spectrum

$$Spec(\overline{G}) = \begin{pmatrix} n - r - 1 & -\lambda_2 - 1 & \dots & -\lambda_k - 1 \\ 1 & m_2 & \dots & m_k \end{pmatrix}$$

**Theorem 1.2. [8]2***If $G$ is a $r$ regular graph with $n$ vertices, then its largest eigenvalue of degree product adjacency matrix is $\alpha_1 = r^3$.*

From Theorem 1.2, the degree product adjacency spectrum of $G$ is,

$$Spec_{DPA}(G) = \begin{pmatrix} r^3 & \alpha_2 & \alpha_3 & \dots & \alpha_k \\ 1 & m_2 & m_3 & \dots & m_k \end{pmatrix}$$

**Theorem 1.3.3[7]***If $K_{m,n} (m = n)$ is a complete bipartite graph. Then the degree product adjacency spectrum of a graph $K_{m,n} (m = n)$ is,*

$$Spec_{DPA}(K_{n,n}) = \begin{pmatrix} n^3 & 0 & \dots & 0 & -n^3 \\ 1 & m_2 & \dots & m_{k-1} & 1 \end{pmatrix}.$$

**Theorem 1.4.[7]***If $K_n$ is a complete graph with n vertices. Then the degree product adjacency spectrum of $K_n$ is*

$$Spec_{DPA}(K_n) = \begin{pmatrix} (n-1)^3 & [-(n-1)^2] \\ 1 & (n-1) \end{pmatrix}.$$

**Remark 1.5.4**[8],

$$Spec_{DPA}(L(G)) = \begin{pmatrix} (2r-2)^3 & (2r-2)^2\left(\frac{\alpha_2}{r^2}+r-2\right) & \dots & -8(r-1)^2 \\ 1 & m_2 & \dots & \frac{n(r-2)}{2} \end{pmatrix}$$

## 2. Main Results

**Theorem 2.1.5***If $G$ is a $r$ regular graph and the adjacency eigenvalue of $G$ are $\lambda_i$ ; $i = 1,2,\dots,k$, then the degree product adjacency eigenvalue for the graph $G$ are $\alpha_i = r^2\lambda_i$ ; $i = 1,2,\dots,k$.*

***Proof.*** Consider the $r$ regular graph $G$ with $n$ vertices where $\alpha_1, \alpha_2, \dots, \alpha_k$ are the eigenvalues of degree product adjacency matrix of $G$.

We prove this Theorem by using the following facts.

**i.** Consider the cycle graph $C_3$ and the adjacency eigenvalues of $C_3$ are $-1, -1, 2$. Now the degree product adjacency eigenvalues of $C_3$ are $-4, -4, 8$.

Here the cycles are 2-regular graphs, then the product of square of regularity and eigenvalues of adjacency matrix are equal to eigenvalues of degree product adjacency matrix i.e., $\alpha_i = r^2\lambda_i$.

And this condition holds for all cycle graphs $C_n$; $n \geq 3$.

**ii.** Now consider the complete graph $K_n$ and its eigenvalues for the adjacency matrix are $n-1$ with multiplicity 1 and $-1$ with multiplicity $n-1$. Now from Theorem 1.4, The eigenvalues of degree product adjacency matrix of $K_n$ are $(n-1)^3$ with multiplicity 1 and $-(n-1)^2$ with multiplicity $(n-1)$.

The complete graph $K_n$ is $(n-1)$ regular, therefore the eigenvalues of degree product adjacency matrix are equal to product of square of regularity and eigenvalues of adjacency matrix of $K_n$.

From these two conditions, it follows that all regular graphs holds the equality i.e., $\alpha_i = r^2\lambda_i$ and also by observation one can conclude that the eigenvalues of $DPA(G)$, where $G$ is regular graph are equal to product of square of regularity and eigenvalues $(\lambda_i;\, i = 1,2,\ldots,k)$ of $A(G)$ i.e., $\alpha_i = r^2\lambda_i$.

**Theorem 2.2. 6***If $G$ is a $r$ regular graph, then*

$$Spec_{DPA}(\overline{G}) = \begin{pmatrix} (n-r-1)^3 & \left[(n-r-1)^2\left(\frac{\alpha_2}{r^2}-1\right)\right] & \ldots & \left[(n-r-1)^2\left(\frac{\alpha_k}{r^2}-1\right)\right] \\ 1 & m_2 & \ldots & m_k \end{pmatrix}.$$

and

$$E_{DPA}(\overline{G}) = (n-r-1)^2\left(2-n-\sum_{i=2}^{n}\frac{\alpha_i}{r^2}\right)$$

***Proof.*** Consider the $r$ regular graph $G$ and the graph $\overline{G}$ is complement of $G$. From Theorem 1.1, the graph $\overline{G}$ is $(n-r-1)$ regular.

Now from Theorem 1.2, the maximum eigenvalue of $DPA(G)$ is $r^3$ for all regular graphs. Hence from Theorem 1.1 and Theorem 2.1, the degree product adjacency spectra of $\overline{G}$ is,

$$Spec_{DPA}(\overline{G}) = \begin{pmatrix} (n-r-1)^3 & \left[(n-r-1)^2\left(\frac{\alpha_2}{r^2}-1\right)\right] & \ldots & (n-r-1)^2\left(\frac{\alpha_k}{r^2}-1\right) \\ 1 & m_2 & \ldots & m_k \end{pmatrix}.$$

By using the spectrum of $DPA(\overline{G})$,

$$E_{DPA}(\overline{G}) = (n-r-1)^2\left(2-n-\sum_{i=2}^{n}\frac{\alpha_i}{r^2}\right)$$

**Theorem 2.3.7***If $G$ is $r$ regular graph but not complete bipartite having the smallest eigenvalue greater than or equal to $r^2(1-r)$, then*

$$E_{DPA}(\overline{L(G)}) \quad = \quad \left(\frac{nr-2(2r-1)}{2}\right)^2 ((r-1)(2n-4)-2)$$

***Proof.*** Consider the $r$ regular graph $G$ with $n$ vertices and is not complete bipartite, then from Theorem 1.2, $r^3 \geq \alpha_2 \geq \alpha_3 \geq \ldots \geq \alpha_k$ are the distinct eigenvalues of $DPA(G)$. Therefore the spectrum of $DPA(G)$ is,

$$Spec_{DPA}(G) = \begin{pmatrix} r^3 & \alpha_2 & \alpha_3 & \ldots & \alpha_k \\ 1 & m_2 & m_3 & \ldots & m_k \end{pmatrix}$$

Now from Remark 1.5, Theorem 2.1 and Theorem 2.2,

$$Spec_{DPA}(\overline{L(G)}) = \begin{pmatrix} \left(\frac{nr-2(2r-1)}{2}\right)^3 & \left[\left(\frac{nr-2(2r-1)}{2}\right)^2\left(\frac{-\alpha_2}{r^2}-r+1\right)\right] & \ldots & \left(\frac{nr-2(2r-1)}{2}\right)^2 \\ 1 & m_2 & \ldots & \frac{n(r-2)}{2} \end{pmatrix}.$$

Since $\frac{-\alpha_i}{r^2} - r + 1 \leq 0$; $i = 2,3,\ldots,k$ is always true, thus

$$\begin{aligned} E_{DPA}(\overline{L(G)}) \quad &= \quad \left(\frac{nr-2(2r-1)}{2}\right)^3 + \left(\frac{nr-2(2r-1)}{2}\right)^2 \sum_{i=2}^{k} m_i \left(\frac{\alpha_i}{r^2} - r + 1\right) \\ &+ \left(\frac{nr-2(2r-1)}{2}\right)^2 \frac{n(r-2)}{2} \\ &= \quad \left(\frac{nr-2(2r-1)}{2}\right)^2 \left(\frac{nr}{2} - 2r + 1 + \frac{nr}{2} - \frac{2n}{2}\right) \\ &+ \left(\frac{nr-2(2r-1)}{2}\right)^2 \left(\sum_{i=2}^{k} \frac{m_i\alpha_i}{r^2} + (r-1)\sum_{i=2}^{k} m_i\right) \end{aligned}$$

From Theorem 1.2 and number of multipilicities in the spectra of $DPA(G)$,

$$\begin{aligned} &r^3 + \textstyle\sum_{i=2}^{k} m_i\alpha_i = 0 \qquad and \qquad 1 + \textstyle\sum_{i=2}^{k} m_i = n \\ &i.e., \qquad \textstyle\sum_{i=2}^{k} \frac{m_i\alpha_i}{r^2} = -r i.e., \qquad \textstyle\sum_{i=2}^{k} m_i = n-1 \end{aligned} \quad (4)$$

By using the equation (4) in $E_{DPA}(\overline{L(G)})$,

$$E_{DPA}(\overline{L(G)}) = \left(\frac{nr-2(2r-1)}{2}\right)^2 (nr-2r-n+1) + \left(\frac{nr-2(2r-1)}{2}\right)^2 (-r+(r-1)(n-1))$$

After simplification,

$$E_{DPA}(\overline{L(G)}) = \left(\frac{nr-2(2r-1)}{2}\right)^2 ((r-1)(2n-4)-2)$$

**Theorem 2.4. 8***If $G$ is a complete bipartite and $r$ regular graph having the second smallest eigenvalue greater than or equal to $r^2(1-r)$, then*

$$E_{DPA}(\overline{L(G)}) \quad = \quad \left(\frac{nr-2(2r-1)^2}{2}\right)^2 ((r-1)(2n-4))$$

***Proof.*** Consider the $r$ regular graph $G$ with $n$ vertices and is complete bipartite graph.

Now from Theorem 1.3 the complete bipartite graph $K_{n,n}$ is $n$ regular then the regularity $r$ of $K_{n,n}$ is $n$ i.e., $r=n$, then $r^3 \geq \alpha_2 \geq \alpha_3 \geq \ldots \geq \alpha_{k-1} \geq -r^3$ are the distinct eigenvalues of $DPA(G)$. Therefore the spectrum of $DPA(G)$ is,

$$Spec_{DPA}(G) = \begin{pmatrix} r^3 & \alpha_2 & \alpha_3 & \ldots & \alpha_{k-1} & -r^3 \\ 1 & m_2 & m_3 & \ldots & m_{k-1} & 1 \end{pmatrix}$$

Now from Remark 1.5, Theorem 2.1 and Theorem 2.2,

$$Spec_{DPA}(\overline{L(G)}) = \begin{pmatrix} \left(\frac{nr-2(2r-1)}{2}\right)^3 & \left[\left(\frac{nr-2(2r-1)}{2}\right)^2 (-\alpha_2 - r + 1)\right] & \ldots & \left(\frac{nr-2(2r-1)}{2}\right)^2 \\ 1 & m_2 & \ldots & \frac{n(r-2)}{2}+1 \end{pmatrix}.$$

Since $\frac{-\alpha_i}{r^2} - r + 1 \leq 0$; $i = 2,3,\ldots,k$ is always true,

thus

$$E_{DPA}(\overline{L(G)}) = \left(\frac{nr-2(2r-1)}{2}\right)^3 + \left(\frac{nr-2(2r-1)}{2}\right)^2 \sum_{i=2}^{k-1} m_i(\alpha_i + r - 1) + \left(\frac{nr-2(2r-1)}{2}\right)^2 \left(\frac{n(r-2)}{2} + 1\right)$$

$$= \left(\frac{nr-2(2r-1)}{2}\right)^2 (nr - 2r + 2 - n) + \left(\sum_{i=2}^{k-1} m_i\alpha_i + \sum_{i=2}^{k-1} m_i(r-1)\right)\left(\frac{nr-2(2r-1)}{2}\right)^2$$

From the spectra of $DPA(K_{n,n})$,

$$r^3 + \sum_{i=2}^{k-1} m_i\alpha_i + (-r^3) = 0 \quad and \quad 1 + \sum_{i=2}^{k-1} m_i + 1 = n$$
$$i.e., \quad \sum_{i=2}^{k-1} m_i\alpha_i = 0 \quad i.e., \quad \sum_{i=2}^{k-1} m_i = n - 2 \qquad (5)$$

By using the equation (5) in $E_{DPA}(\overline{L(G)})$,

$$E_{DPA}(\overline{L(G)}) = \left(\frac{nr-2(2r-1)}{2}\right)^2 (nr - 2r + 2 - n) + \left(\frac{nr-2(2r-1)}{2}\right)^2 (0 + (r-1)(n-2))$$

After simplification,

$$E_{DPA}(\overline{L(G)}) = \left(\frac{nr-2(2r-1)}{2}\right)^2 ((r-1)(2n-4))$$

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