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ADJACENCY AND SEIDEL POLYNOMIAL OF SPLICE AND LINK OF CERTAIN GRAPHS

Harishchandra S. Ramane[1], Ashoka K[2], B. Parvathalu[3] and Daneshwari Patil[4]

[1,2,4]*Department of Mathematics, Karnatak University, Dharwad - 580003, India*

[3]*Department of Mathematics, Karnatak University's Karnatak Arts College, Dharwad - 580001, India*

Email :[1]hsramane@yahoo.com,[2]ashokagonal@gmail.com, [3]bparvathalu@gmail.com, [4]daneshwarip@gmail.com

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Abstract

The adjacency and the Seidel polynomial of a graph are respectively the characteristic polynomial of adjacency and Seidel matrix associated to that graph. The adjacency polynomial of splice and link graphs of some well-known graph classes have been obtained recently in the literature. The equitable partition of a graph plays an important role in finding partial spectrum of a graph. In this article we study the adjacency polynomial of complement of splice and link graphs of certain graphs and also the Seidel polynomial by using the concept of equitable partition.

Keywords: Adjacency polynomial, Seidel polynomial, splice, link, equitable partition.

2010 AMS classification: 05C50

1. Introduction

The adjacency and Seidel polynomial of a graph are respectively the characteristic polynomial of adjacency and Seidel matrix associated to graph. The Energy of a graph is introduced by Ivan Gutman in 1978 [6], defined as the sum of absolute values of the eigenvalues of adjacency matrix. It has become an important area of research in mathematics as well as in chemical graph theory.

A graph $G = (V(G), E(G))$, where $V(G) = \{v_1, v_2, \ldots, v_n\}$ is a non-empty set of elements called vertices, and $E(G)$ is the set of unordered pairs of distinct vertices called edges. Graphs considered here are simple and undirected. We follow [3, 4] for terminology and notations. Given a graph, there are uniquely associated square matrices which store the information about its vertices and the interconnections between them. The aim of spectral graph theory is to study how the eigenvalues and eigenvectors of such a matrix representation of a graph are related to the graph structure. In literature, there are a wide variety of well-known matrices associated to a graph, such as adjacency matrix, Seidel matrix, Laplacian matrix, normalized Laplacian matrix, distance matrix etc.

The adjacency matrix $A(G)$ of a graph G of order n is an$n \times n$ matrix indexed by $V(G)$, whose $(i, j)^{th}$ entry is defined as $a_{ij} = 1$ if $v_i v_j \in E(G)$ and 0 otherwise. The adjacency polynomial of G is defined by $|xI - A(G)|$. The eigenvalues of G are the roots of the adjacency polynomial. The collection of the eigenvalues of the adjacency matrix of a graph is called the adjacency spectra of $G$and it is denoted by Spec $(A(G))$. The complement $G^c$of a graph G has same vertices as G, but two vertices are adjacent in G^c if and only if they are not adjacent in G and $A(G^c) = J - I - A(G)$, where J is the matrix whose all entries are equal to 1 and$I$ is an identity matrix of suitable order. The Seidel matrix [7] of a graph G is an $n \times n$ real symmetric matrix $S(G) = [s_{ij}]$, where $s_{ij} = -1$ if the vertices v_I and $v_j$are adjacent, $s_{ij} = 1$ if the vertices v_i and $v_j$are not adjacent and $s_{ij} = 0$ if $i = j$.It is easy to see that $S(G) = J - I - 2A(G)$. The characteristic polynomial of $S(G)$ is defined as $|xI - S(G)|$ is called Seidel polynomial. The Seidel eigenvalues of $S(G)$ are the roots of the Seidel polynomial. The collection of the eigenvalues of the Seidel matrix of a graph is called the Seidel spectra of G [1] and it is denoted by Spec$(S(G))$. As usual, we denote complete graph by K_n, star graph by S_n and complete bipartite graph by $K_{r,s}$.The spectrum of these graphs is well known in the literature [4].

One of the methods of generating new graphs is to make use of the graph operations. There is a large and increasing number of graph operations such as join, corona, cartesian product, union, tensor product, splice, link etc. The symmetry and regularity are two important and desired properties in many areas including graphs. In many molecular graphs, we have a pointwise symmetry, that is the graph corresponding to the molecule under investigation has two identical subgraphs which are symmetrical at a vertex and it can be done by using one of the graph operations such as splice, link of graphs [2]. In this paper we are motivated to study the adjacency and the Seidel characteristic polynomials of certain class of graphs obtained from the graph operations splice and link through equitable partition.

2. Preliminaries

Definition 2.1 [9] If G is a labeled graph of order n, then the graph $G[G_1, G_2, \dots, G_n]$ is called generalized compositionobtained by taking the disjoint graphs $G_1, G_2, \dots, G_n$ and then joining every vertex of $G_i$to every vertex of G_j whenever $v_i$adjacent to v_j in G.

Definition 2.2 [9] Let $\pi = (V_1, V_2, \dots, V_k)$ be a vertex partition of a graph G. The partition π is called an equitable partition if for all i and j, the number d_{ij} of edges from any vertex in V_i to the cell V_j is independent of the choice of vertex in V_i.

Let $\pi = (V_1, V_2, \dots, V_k)$ be an equitable partition of the vertexset $V(G)$ of a graph G. The quotient graph G/π of G with respect to π is a directed graph with the cells of π as its vertex set, and with d_{ij} arcs from a vertex V_i to a vertex V_j. The adjacency matrix $A(G/\pi)$ of the quotient graph G/π is $k \times k$ matrix whose (i,j)thentry is d_{ij}. Let $\phi(M) = \phi(M; x)$ denotes the characteristic polynomial of matrix M.

Theorem 2.1 [9] If $\pi = (V_1, V_2, \dots, V_k)$ is an equitable partition of graph G then $\phi(A(G/\pi))$ divides$\phi(A(G))$.

Theorem 2.2 [9] If $G_1, G_2, \dots, G_n$ are all regular, then $V(G_1) \cup V(G_2) \cup \dots \cup V(G_n)$ is anequitable partition of $G[G_1, G_2, \dots, G_n]$. And $\phi(G[G_1, G_2, \dots, G_n]) = \phi(G/\pi) \prod_{i=1}^{n} \phi(G_i)/(x - r_i)$.

Tomislav Došlić[10] defined splice and link of two graphs in the year 2005.

Definition 2.3 [10] Let G_1, G_2 be two graphs and let us label two vertices, one in $V(G_1)$ and the other in $V(G_2)$ by v. The vertex joining graph at v or the splice of these two graphs is denoted by $G_1 \vee_v G_2$ and obtained by identifying the vertices v of the two graphs.

Definition 2.4 [10] Let G_1, G_2be two graphs and let us label two vertices, one in $V(G_1)$ and the other in $V(G_2)$, by v. The edge joining graph at v or the link of these two graphs is denoted by $G_1 \vee^e{}_v\, G_2$ and obtained by adding a new edge between the identified vertices v of two graphs.

Proposition 2.1 (Schur Complement)[1] Suppose that the order of all four matrices M, N, P and Q satisfy the rules of operations on matrices. Then we have,

$$det\begin{pmatrix} M & N \\ P & Q \end{pmatrix} = \begin{cases} \det(Q)\det(M - NQ^{-1}P), \text{if } Q \text{ is a non singular matrix,} \\ \det(M)\det(Q - PM^{-1}N), \text{if } M \text{ is a non singular matrix.} \end{cases}$$

Theorem 2.2 [11] If a graph G has an equitable partition π, then it is also equitable for the complement G^c. The corresponding quotient adjacency matrix is given by $A(G^c/\pi) = J(\pi) - A(G/\pi) - I$, where I is the $k \times k$ identity matrix and $J(\pi) = \begin{pmatrix} n_1 & \cdots & n_k \\ \vdots & \ddots & \vdots \\ n_1 & \cdots & n_k \end{pmatrix}$ and $Spec(A(G^c)) = Spec(A(G^c/\pi)) \cup \{-1 - [Spec\big(A(G)\big)\backslash Spec(A(G/\pi))]\}$.

The following proposition follows from Theorem 2.2 easily.

Proposition 2.3 If a graph G has an equitable partition π, it is also equitable for the Seidel matrix $S(G)$. The corresponding quotient Seidel matrix is given by $S(G/\pi) = J(\pi) - 2A(G/\pi) - I$, where I is the $k \times k$ identity matrix and $J(\pi) = \begin{pmatrix} n_1 & \cdots & n_k \\ \vdots & \ddots & \vdots \\ n_1 & \cdots & n_k \end{pmatrix}$.

Theorem 2.4 [5]Let H be a graph of order n. Let $G = H[G_1, G_2, \ldots, G_n]$, where G_i is anr_i regular graph of order n_i then

$$Spec\,(S(G)) = Spec(S(G/\pi)) \cup [\cup_{k=1}^{n} (Spec(G_k)\backslash\{n_k - 2r_k - 1\})].$$

Theorem 2.5 [1] Let G be an r-regular graph on n vertices. If$r, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of G, then its Seidel eigenvalues are $n - 1 - 2r$ and $-1 - 2\lambda_i$, $i = 2, 3, \ldots, n$.

3. Adjacency and Seidel polynomial of splice and link of graphs

In [8] Ramane etal. generalized the concept of splice and link of two graphs to p copies of simple and connected graphs $G_1, G_2, \ldots, G_p$ as follows.

Definition 3.1 [8] Let $G_1, G_2, \dots, G_p$be p graphs and let us label p vertices, one in each $V(G_i)$ for $i = 1, 2, \dots, p$ by v. The vertex joining graph at v or the splic**e** of these graphs be denoted as $\vee_v[G_1, G_2, \dots, G_p]$ which is obtained by identifying the vertices v of the p graphs.

The vertex set of $\vee_v[G_1, G_2, \dots, G_p]$ is $V(\vee_v[G_1, G_2, \dots, G_p]) = V(G_1) \cup V(G_2) \cup \cdots \cup V(G_p)$ and the edge set is $E(\vee_v[G_1, G_2, \dots, G_p]) = E(G_1) \cup E(G_2) \cup \cdots \cup E(G_p)$.

If $|V(G_1)| = n_1, |V(G_2)| = n_2, \dots |V(G_p)| = n_p$, then $|V(\vee_v[G_1, G_2, \dots, G_p])| = n_1 + n_2 + \cdots + n_p - (p-1)$and if $|E(G_1)| = m_1$, $|E(G_2)| = m_2, \dots |E(G_p)| = m_p$then$|E(\vee_v[G_1, G_2, \dots, G_p])| = m_1 + m_2 + \dots + m_p$. See the following Fig. 1.

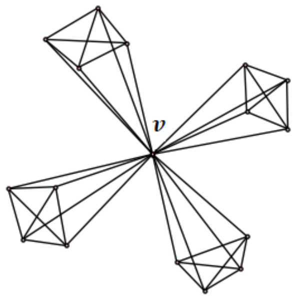

Fig.1:$\vee_v[K_5, K_5, K_5, K_5]$

Definition 3.2 [8] Let $G_1, G_2, \dots, G_{2p}$ be $2p$ graphs and let us label p vertices, one in each $V(G_i)$for $i = 1, 2, \dots, p$ by v and other p vertices, one in each $V(G_i)$ for $i = p+1, p+2, \dots, 2p$ by v'. The edge joining graph at vv' or the link of these graphs is denoted by$\vee^e_{vv'}[G_1, G_2, \dots, G_{2p}]$ which is obtained by adding a new edge between the identified vertices v and v' of $2p$ graphs (for the purpose of symmetry in the structure, p copies are taken at v and other p copies at v').

The vertex set of $\vee^e_{vv'}[G_1, G_2, \dots, G_{2p}]isV(\vee^e_{vv'}[G_1, G_2, \dots, G_{2p}])$=$V(G_1) \cup V(G_2) \cup \cdots \cup V(G_{2p})$and the edge set is $E(\vee^e_{vv'}[G_1, G_2, \dots, G_{2p}]) = E(G_1) \cup E(G_2) \cup \cdots \cup E(G_{2p}) \cup \{e\}$. See the following Fig. 2.

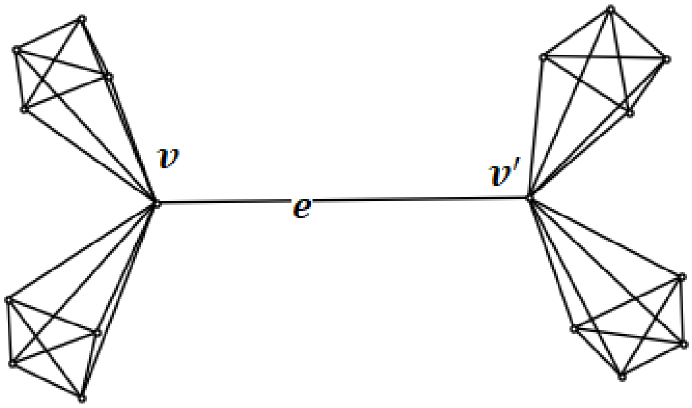

Fig. 2: $\vee^{e}_{vv'}[K_5, K_5, K_5, K_5]$

Theorem 3.1: The adjacency polynomial of complement of $\vee_v$ $[K_n, K_n, \ldots, K_n]$ (p copies of K_n) and the Seidel polynomial of $\vee_v[K_n, K_n, \ldots, K_n]$ (p copies of K_n) are, $(x-n+2)^{p-1}$ $(x)^{p(n-2)}(x^2-[p(n-1)-(n-2)]x)$and $(x+2n-3)^{p-1}$ $(x-1)^{p(n-2)}(x^2-[p(n-1)-2n+3]x-p(n-1))$ respectively.

Proof: Making the vertices of $\vee_v$ $[K_n, K_n, \ldots, K_n]$ as two partite sets: $V_1=\{v\}$ and $V_2=\{u: u \text{ is adjacent to } v\}$, these two partite sets lead to the quotient matrix

$$A(G/\pi)=\begin{pmatrix}0 & p(n-1)\\ 1 & n-2\end{pmatrix}.$$

Using Theorem 2.2, the adjacency quotient matrix of complement of $\vee_v$ $[K_n, K_n, \ldots, K_n]$ is $A\left(\frac{G^c}{\pi}\right)=J(\pi)-A\left(\frac{G}{\pi}\right)-I=\begin{pmatrix}0 & 0\\ 0 & (n-1)(p-1)\end{pmatrix}$, where $J(\pi)=\begin{pmatrix}1 & p(n-1)\\ 1 & p(n-1)\end{pmatrix}$. From the characteristic polynomial of $A(G^c/\pi)$ we have, $\phi(G^c/\pi)=(x^2-[p(n-1)(n-2)]x)$ and by Theorem 2.2 the remaining part of spectrum of complement of $\vee_v$ $[K_n, K_n, \ldots, K_n]$ is $\{(1-n)^{(p-1)}, 0^{p(n-2)}\}$. Now, by usingProposition 2.3, the Seidel quotient matrix of $\vee_v$ $[K_n, K_n, \ldots, K_n]$ is $S(G/\pi)=\begin{pmatrix}0 & -p(n-1)\\ -1 & p(n-1)-2n+3\end{pmatrix}$ with its characteristic polynomial $\phi(S(G/\pi))=(x^2-[p(n-1)-2n+3]x-p(n-1))$ and from Theorem 2.4 the remaining part of the Seidel spectrum of $\vee_v$ $[K_n, K_n, \ldots, K_n]$ is $\{(-2n+3)^{(p-1)}, 1^{p(n-2)}\}$. Hence the result follows.

Theorem 3.2: The adjacency polynomial of complement of $\vee^{e}{}_{vv'}[K_n, K_n, \ldots, K_n]$(2p copies of K_n) and the Seidel polynomial of $\vee^{e}{}_{vv'}[K_n, K_n, \ldots, K_n]$ (2p copies of K_n) are$x^{2p(n-2)}\big(x-(1-n)\big)^{2p-2}(x^4-2(n-1)(p-1)x^3+(n^2-2pn(n-1)-2n+1)x^2+2p(n-1)^2(p-1)x+p^2(n-1)^2)$ and $(x-1)^{2p(n-2)}\big(x-(3-2n)\big)^{2p-2}(x+1)(x-\big(2p(n-1)+(3-2n)\big)\big(x^2+(2n-4)x-4p(n-1)+(3-2n)\big)$ respectively.

Proof: From $\vee^{e}{}_{vv'}[K_n, K_n, \ldots, K_n]$ we have, $e = vv'$ and p copies of K_n identified at v and other p copies of K_n identified at v'. Making $(2np-2p+2)$ vertices of $\vee^{e}{}_{vv'}[K_n, K_n, \ldots, K_n]$ into four partite sets: $V_1 = \{v\}$, $V_2 = \{v'\}$, $V_3 = \{u: u \text{ is adjacent to } v\}$ and $V_4 = \{u': u' \text{ is adjacent to } v'\}$ these four partite sets lead to the quotient matrix $A(G/\pi) = \begin{pmatrix} 0 & 1 & p(n-1) & 0 \\ 1 & 0 & 0 & p(n-1) \\ 1 & 0 & n-2 & 0 \\ 0 & 1 & 0 & n-2 \end{pmatrix}$.

UsingTheorem2.2,the adjacency quotient matrix of complement of $\vee^{e}{}_{vv'}[K_n, K_n, \ldots, K_n]$ is

$$A(G^c/\pi) = J(\pi) - A(G/\pi) - I = \begin{pmatrix} 0 & 0 & 0 & p(n-1) \\ 0 & 0 & p(n-1) & 0 \\ 0 & 1 & (n-1)(p-1) & p(n-1) \\ 1 & 0 & p(n-1) & (n-1)(p-1) \end{pmatrix},$$

where $J(\pi) = \begin{pmatrix} 1 & 1 & p(n-1) & p(n-1) \\ 1 & 1 & p(n-1) & p(n-1) \\ 1 & 1 & p(n-1) & p(n-1) \\ 1 & 1 & p(n-1) & p(n-1) \end{pmatrix}$.

By Proposition 2.1, we have $\phi(G^c/\pi) = x^4-2(n-1)(p-1)x^3+(n^2-2pn(n-1)-2n+1)x^2+2p(n-1)^2(p-1)x+p^2(n-1)^2$ and by Theorem 2.2 theremaining part of the spectrum of complement of $\vee^{e}{}_{vv'}[K_n, K_n, \ldots, K_n]$ is $\{0^{2p(n-2)}, (1-n)^{(2p-2)}\}$. Now, by using Proposition 2.3 the Seidel quotient matrix of $\vee^{e}{}_{vv'}[K_n, K_n, \ldots, K_n]$ is

$$S(G/\pi) = \begin{pmatrix} 0 & -1 & -p(n-1) & p(n-1) \\ -1 & 0 & p(n-1) & -p(n-1) \\ -1 & 1 & p(n-1)+(3-2n) & p(n-1) \\ 1 & 0 & p(n-1) & p(n-1)+(3-2n) \end{pmatrix}.$$

By Proposition 2.1, we have $\phi(S(G/\pi)) = (x+1)(x-\big(2p(n-1)+(3-2n)\big)\big(x^2+(2n-4)x-4p(n-1)+(3-2n)\big)$ and from Theorem 2.4 the remaining part of the Seidel spectrum of $\vee^{e}{}_{vv'}[K_n, K_n, \ldots, K_n]$is $\{1^{2p(n-2)}, (3-2n)^{(2p-2)}\}$. Hence the result follows.

Theorem 3.3: The adjacency polynomial of complement of $\vee_v[S_n,\ S_n]$ (2 copies of S_n)and the Seidel polynomial of $\vee_v[S_n,\ S_n]$ (2 copies of S_n) are $(x+1)^{2n-6}(x^5-(2n-6)x^4-(6n-12)x^3+(2n^2-10n+10)x^2+(n^2-5)x-(2n^2-10n+12))$ and $(x+1)^{2n-6}(x^5-(2n-6)x^4-(8n-14)x^3+(8n^2-28n+16)x^2+(24n-55)x-(24n^2-110n+126)$ respectively.

Proof: Making the vertices of $\vee_v[S_n,\ S_n]$ which is obtained from 2 copies of the star S_n as five partite sets:
$V_1=\{u: u \text{ is the central vertex of the } 1^{\text{th}} \text{ copy of the star } S_n\}$,
$V_2=\{u': u' \text{ is the central vertex of the } 2^{\text{nd}} \text{ copy of the star } S_n\}$,$V_3=\{v\}$,$V_4=\{w: w \text{ is noncentral vertex of the } 1^{\text{th}} \text{ copy of the star } S_n \text{ such that } w\neq v\}$ and $V_5=\{w': w' \text{is noncentral vertex of the } 2^{\text{nd}} \text{ copy of the star } S_n \text{ such that } w'\neq v\}$ these five partite sets lead to the quotient matrix

$$A(G/\pi)=\begin{pmatrix}0&0&1&n-2&0\\0&0&1&0&n-2\\1&1&0&0&0\\1&0&0&0&0\\0&1&0&0&0\end{pmatrix}.$$

Using Theorem 2.2, the adjacency quotient matrix of complement of $\vee_v[S_n,\ S_n]$ is

$$A(G^c/\pi)=J(\pi)-A(G/\pi)-I=\begin{pmatrix}0&1&0&0&n-2\\1&0&0&n-2&0\\0&0&0&n-2&n-2\\0&1&1&n-3&n-2\\1&0&1&n-2&n-3\end{pmatrix},$$

where $J(\pi)=\begin{pmatrix}1&1&1&n-2&n-2\\1&1&1&n-2&n-2\\1&1&1&n-2&n-2\\1&1&1&n-2&n-2\\1&1&1&n-2&n-2\end{pmatrix}$.

By Proposition 2.1, we have $\phi(G^c/\pi)=x^5-(2n-6)x^4-(6n-12)x^3+(2n^2-10n+10)x^2+(n^2-5)x-(2n^2-10n+12)$ and by Theorem 2.2 remaining part of the spectrum of complement of $\vee_v[S_n,\ S_n]$ is $\{-1^{2n-6}\}$. Now, by using Proposition 2.3, the Seidel quotient matrix of $\vee_v[S_n,\ S_n]$ is

$$S(G/\pi) = \begin{pmatrix} 0 & 1 & -1 & -(n-2) & n-2 \\ 1 & 0 & -1 & n-2 & -(n-2) \\ -1 & -1 & 0 & n-2 & n-2 \\ -1 & 1 & 1 & n-3 & n-2 \\ 1 & -1 & 1 & n-2 & n-3 \end{pmatrix}.$$

By Proposition 2.1, we have $\phi(S(G/\pi)) = x^5 - (2n-6)x^4 - (8n-14)x^3 + (8n^2 - 28n + 16)x^2 + (24n - 55)x - (24n^2 - 110n + 126)$ and from Theorem 2.4 the remaining part of the Seidel spectrum of $\vee_v[S_n, S_n]$ is $\{-1^{2n-6}\}$. Hence the result follows.

Theorem 3.4: The adjacency polynomial of complement of $\vee^e{}_{vv'}[S_n, S_n]$ (2 copies of S_n) and the Seidel polynomial of $\vee^e{}_{vv'}[S_n, S_n]$ (2 copies of S_n) are

$$\phi(\overline{\vee^e{}_{vv'}[S_n, S_n]}: x) = \begin{cases} (x+1)^{2n-4}\begin{pmatrix} x^4 - (2n-4)x^3 - (4n-5)x^2 \\ +(2n^2 - 6n + 4)x + (n^2 - 2n + 1) \end{pmatrix}, \\ \qquad v \text{ and } v' \text{are the central vertices,} \\ (x+1)^{2n-6}\begin{pmatrix} x^6 - (2n-6)x^5 - (8n-14)x^4 + (2n^2 - 12n + 10)x^3 \\ +(3n^2 - 4n - 6)x^2 \\ -(2n^2 - 12n + 14)x - (2n-5) \end{pmatrix}, \\ \qquad v \text{ and } v' \text{are the non central vertices.} \end{cases}$$

and

$$\phi(S(\vee^e{}_{vv'}[S_n, S_n]): x) = \begin{cases} (x+1)^{2n-4}\begin{pmatrix} x^4 - (2n-4)x^3 - (6n-6)x^2 + (8n^2 - 22n + 12)x \\ +(18n^2 - 18n + 9) \end{pmatrix}, \\ \qquad v \text{ and } v' \text{are the central vertices,} \\ (x+1)^{2n-6}(x^6 - (2n-6)x^5 - (11n-17)x^4 + 4(2n^2 - 5n - 1)x^3 \\ +(8n^2 + 18n - 69)x^2 - 2(20n^2 - 79n + 73)x + \\ (24n^2 - 143n + 195), \\ \qquad v \text{ and } v' \text{are the non central vertices.} \end{cases}$$

Proof: (i) When v and v' are central vertices of two copies of S_n respectively, making $2n$ vertices of $\vee^e{}_{vv'}[S_n, S_n]$ which includes two copies of S_n into four partite sets: $V_1 = \{v\}$, $V_2 = \{v'\}$, $V_3 = \{u: u \text{ is adjacent to } v\}$ and $V_4 = \{u': u' \text{ is adjacent to } v'\}$ these four partite sets lead to the quotient matrix

$$A(G/\pi)=\begin{pmatrix}0 & 1 & n-1 & 0\\ 1 & 0 & 0 & n-1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}.$$

Using Theorem 2.2, the adjacency quotient matrix of complement of $\vee^{e}{}_{vv'}[S_n,\, S_n]$is given by $A\left(\frac{G^c}{\pi}\right)=J(\pi)-A\left(\frac{G}{\pi}\right)-I=\begin{pmatrix}0 & 0 & 0 & n-1\\ 0 & 0 & n-1 & 0\\ 0 & 1 & n-2 & n-1\\ 1 & 0 & n-1 & n-2\end{pmatrix}$,

where $J(\pi)=\begin{pmatrix}1 & 1 & n-1 & n-1\\ 1 & 1 & n-1 & n-1\\ 1 & 1 & n-1 & n-1\\ 1 & 1 & n-1 & n-1\end{pmatrix}$.

ByProposition 2.1, we have $\phi(G^c/\pi)=(x^4-(2n-4)x^3-(4n-5)x^2+(2n^2-6n+4)x+(n^2-2n+1)$ and by Theorem 2.2 remaining part of the spectrum of complement of $\vee^{e}{}_{vv'}[S_n,\, S_n]$ is $\{-1^{2n-4}\}$. Now, by using Proposition 2.3, the Seidel quotient matrix of $\vee^{e}{}_{vv'}[S_n,\, S_n]$is given by

$$S(G/\pi)=\begin{pmatrix}0 & -1 & -(n-1) & (n-1)\\ -1 & 0 & (n-1) & -(n-1)\\ -1 & 1 & (n-2) & (n-1)\\ 1 & -1 & (n-1) & (n-2)\end{pmatrix}.$$

By Proposition 2.1, we have

$\phi(S(G/\pi))=x^4-(2n-4)x^3-(6n-6)x^2+(8n^2-22n+12)x+$
$(18n^2-18n+9)$

and from Theorem 2.4 the remaining part of the Seidel spectrum of $\vee^{e}{}_{vv'}[S_n,\, S_n]$ is $\{-1^{2n-4}\}$.

(ii) When v and v' are pendent vertices two copies of S_n, making 2n vertices of $\vee^{e}{}_{vv'}[S_n,\, S_n]$ which includes two copies of S_n into six partite sets: $V_1=\{\,v\}$,
$V_2=\{u'\colon u'\text{is a pendent vertex of}\,S_n\text{ which is adjacent to }v'\text{and is such that }e=uu'\}$,$V_3=\{w'\colon w'\text{is adjacent to }v'\text{with }w'\neq u'\}$, $V_4=\{\,u\colon u\text{ is a pendent vertex of }S_n\text{which is adjacent to }v\text{ and is such that }e=uu'\}$,
$V_5=\{\,w\colon w\text{ is adjacent to }v\text{ with }w\neq u\,\}$ and $V_6=\{v'\}$, these six partite sets lead to the quotient matrix $A(G/\pi)=\begin{pmatrix}0 & 0 & 0 & 1 & n-2 & 0\\ 0 & 0 & 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 1\\ 1 & 1 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & n-2 & 0 & 0 & 0\end{pmatrix}$.

Using Theorem 2.2, the adjacency quotient matrix of complement of $\vee^{e}{}_{vv'}[S_n,\, S_n]$is

$$A\left(\frac{G^c}{\pi}\right) = J(\pi) - A\left(\frac{G}{\pi}\right) - I = \begin{pmatrix} 0 & 1 & n-2 & 0 & 0 & 1 \\ 1 & 0 & n-2 & 0 & n-2 & 0 \\ 1 & 1 & n-3 & 1 & n-2 & 0 \\ 0 & 0 & n-2 & 0 & n-2 & 1 \\ 0 & 1 & n-2 & 1 & n-3 & 1 \\ 1 & 0 & 0 & 1 & n-2 & 0 \end{pmatrix}$$,where

$$J(\pi) = \begin{pmatrix} 1 & 1 & n-2 & 1 & n-2 & 1 \\ 1 & 1 & n-2 & 1 & n-2 & 1 \\ 1 & 1 & n-2 & 1 & n-2 & 1 \\ 1 & 1 & n-2 & 1 & n-2 & 1 \\ 1 & 1 & n-2 & 1 & n-2 & 1 \\ 1 & 1 & n-2 & 1 & n-2 & 1 \end{pmatrix}.$$

By Proposition 2.1, we have

$\phi(G^c/\pi) = (x^6 - (2n-6)x^5 - (8n-14)x^4 + (2n^2 - 12n + 10)x^3 + (3n^2 - 4n - 6)x^2 - (2n^2 - 12n + 14)x - (2n-5))$ and by Theorem 2.2 remaining part of the spectrum of complement of $\vee^e_{vv'}[S_n, S_n]$ is $\{-1^{2n-6}\}$. Now, by using Proposition 2.3, the Seidel quotient matrix of $\vee^e_{vv'}[S_n, S_n]$is

$$S(G/\pi) = \begin{pmatrix} 0 & 1 & n-2 & -1 & -(n-2) & 1 \\ 1 & 0 & n-2 & -1 & n-2 & -1 \\ 1 & 1 & n-3 & 1 & n-2 & -1 \\ -1 & -1 & n-2 & 0 & n-2 & 1 \\ -1 & 1 & n-2 & 1 & n-3 & 1 \\ 1 & -1 & -(n-2) & 1 & n-2 & 0 \end{pmatrix}.$$

By Proposition 2.1, we have

$\phi(S(G/\pi)) = x^6 - (2n-6)x^5 - (11n-17)x^4 + 4(2n^2 - 5n - 1)x^3 + (8n^2 + 18n - 69)x^2 - 2(20n^2 - 79n + 73)x + (24n^2 - 143n + 195)$andfrom Theorem 2.4 the remaining part of the Seidel spectrum of $\vee^e_{vv'}[S_n, S_n]$ is $\{-1^{2n-6}\}$. Hence the result follows.

Theorem 3.5: The adjacency polynomial of complement of $\vee_v[K_{r,s}, K_{r,s}]$ (2 copies of $K_{r,s}$)and the Seidel polynomial of $\vee_v[K_{r,s}, K_{r,s}]$(2 copies of $K_{r,s}$), where $r < s$ are

$$\phi(\curlyvee_v[K_{r,s},K_{r,s}]:\ x) = \begin{cases} (x+1)^{2r+2s-6}(x^5-(2r+2s-6)x^4-(8r-2rs+8s-14)x^3 \\ \quad -(12r-rs^2+s^2-r^2s-6rs+13s-16)x^2 \\ \quad -(8r+r^2s^2-3rs^2+2s^2-2r^2s-6rs+10s-9)x \\ \quad -(2r+2r^2s^2-4rs^2+2s^2-r^2s-2rs+3s-2)) \\ \quad \textit{if } v \text{ is selected among the r vertices of } K_{r,s}, \\ (x+1)^{2r+2s-6}(x^5-(2r+2s-6)x^4-(8s-2rs+8r-14)x^3 \\ \quad -(12s-sr^2+r^2-s^2r-6rs+13r-16)x^2 \\ \quad -(8s+r^2s^2-3sr^2+2r^2-2s^2r-6rs+10r-9)x \\ \quad -(2s+2r^2s^2-4sr^2+2r^2-s^2r-2rs+3r-2)), \\ \quad \textit{if } v \text{ is selected among the s vertices of } K_{r,s} \end{cases}$$

With $\phi(\curlyvee_v[K_{r,s},K_{r,s}]:\ x)=(x+1)^{2r+2s-6}(x^5-(4r-6)x^4-(8r-2r^2+8r-14)x^3-(12r-21r^3-5r^2+13r-16)x^2-(8r+r^4-5r^3-4r^2+10r-9)x-(2r+2r^4-4r^2-r^3+3r-2))$ when $r=s$

and

$$\phi(\curlyvee_v[K_{r,s},K_{r,s}]:\ x) = \begin{cases} (x+1)^{2r+2s-6}(x^5-(2r+2s-6)x^4-(8r+6s-14)x^3 \\ \quad -(12r-8rs^2+8s^2-8r^2s+12rs+2s-16)x^2 \\ \quad -(8r+16r^2s^2-40rs^2+24s^2-16r^2s+24rs-6s-9)x \\ \quad -(2r+16r^2s^2-32rs^2+16s^2-8r^2s+12rs-4s-2)) \\ \quad \textit{if } v \text{ is selected among the r vertices of } K_{r,s}, \\ (x+1)^{2r+2s-6}(x^5-(2r+2s-6)x^4-(8s+6r-14)x^3 \\ \quad -(12s-8sr^2+8r^2-8s^2r+12rs+2r-16)x^2 \\ \quad -(8s+16r^2s^2-40sr^2+24r^2-16s^2r+24rs-6r-9)x \\ \quad -(2s+16r^2s^2-32sr^2+16r^2-8s^2r+12rs-4r-2)), \\ \quad \textit{if } v \text{ is selected among the s vertices of } K_{r,s} \end{cases}$$

with $\phi(\curlyvee_v[K_{r,s},K_{r,s}]:\ x)=(x+1)^{2r+2s-6}(x^5-(4r-6)x^4-(14r-14)x^3-(-16r^3+20r^2+14r-16)x^2-(16r^4-56r^3+48r^2+2r-9)x-(16r^4+28r^2-40r^3-2r-2))$ when $r=s$.

Proof: (i) when v is selected among r vertices of $K_{r,s}$, making the vertices of $\curlyvee_v[K_{r,s},K_{r,s}]$ which includes two copies of $K_{r,s}$ into five partite sets: $V_1=\{v\}$,
$V_2=\{u: u$ is not adjacent to v in first copy of $K_{r,s}\}$,
$V_3=\{u': u'$ is not adjacent to v second copy of $K_{r,s}\}$,

$V_4 = \{w: w$ is adjacent to v in first copy of $K_{r,s}\}$ and

$V_5 = \{w': w'$is adjacent to v insecond copy of $K_{r,s}\}$. These partite sets lead to the quotient matrix $A(G/\pi) = \begin{pmatrix} 0 & 0 & 0 & s & s \\ 0 & 0 & 0 & s & 0 \\ 0 & 0 & 0 & 0 & s \\ 1 & r-1 & 0 & 0 & 0 \\ 1 & 0 & r-1 & 0 & 0 \end{pmatrix}$. UsingTheorem 2.2, the adjacency quotient matrix of complement of $\vee_v[K_{r,s}, K_{r,s}]$is

$A\left(\frac{G^c}{\pi}\right) = J(\pi) - A\left(\frac{G}{\pi}\right) - I = \begin{pmatrix} 0 & r-1 & r-1 & 0 & 0 \\ 1 & r-2 & r-1 & 0 & s \\ 1 & r-1 & r-2 & s & 0 \\ 0 & 0 & r-1 & s-1 & s \\ 0 & r-1 & r-1 & s & s-1 \end{pmatrix}$,where

$J(\pi) = \begin{pmatrix} 1 & r-1 & r-1 & s & s \\ 1 & r-1 & r-1 & s & s \\ 1 & r-1 & r-1 & s & s \\ 1 & r-1 & r-1 & s & s \\ 1 & r-1 & r-1 & s & s \end{pmatrix}$.

By Proposition 2.1, we have

$\phi(G^c/\pi) = x^5 - (2r + 2s - 6)x^4 - (8r - 2rs + 8s - 14)x^3 - (12r - rs^2 + s^2 - r^2s - 6rs + 13s - 16)x^2 - (8r + r^2s^2 - 3rs^2 + 2s^2 - 2r^2s - 6rs + 10s - 9)x - (2r + 2r^2s^2 - 4rs^2 + 2s^2 - r^2s - 2rs + 3s - 2)$and by Theorem 2.2 remaining part of the spectrum of complement of $\vee_v[K_{r,s}, K_{r,s}]$is $\{-1^{2r+2s-6}\}$. Now, by using Proposition 2.3, the Seidel quotient matrix of $\vee_v[K_{r,s}, K_{r,s}]$is

$S(G/\pi) = \begin{pmatrix} 0 & r-1 & r-1 & -s & s \\ 1 & r-2 & r-1 & -s & s \\ 1 & r-1 & r-2 & s & -s \\ 0 & -(r-1) & r-1 & s-1 & s \\ 0 & r-1 & -(r-1) & s & s-1 \end{pmatrix}$.

By Proposition 2.1, we have

$\phi(S(G/\pi)) = x^5 - (2r + 2s - 6)x^4 - (8r + 6s - 14)x^3 - (12r - 8rs^2 + 8s^2 - 8r^2s + 12rs + 2s - 16)x^2 - (8r + r^2s^2 - 3rs^2 + 2s^2 - 2r^2s - 6rs + 10s - 9)x - (2r + 2r^2s^2 - 4rs^2 + 2s^2 - r^2s - 2rs + 3s - 2)$ and from Theorem 2.4 the remaining part of the Seidel spectrum of $\vee_v[K_{r,s}, K_{r,s}]$ is $\{-1^{2r+2s-6}\}$. Hence the result follows.

(ii) when v is selected among s vertices of $K_{r,s}$, making the vertices of $\vee_v[K_{r,s}, K_{r,s}]$ which includes two copies of $K_{r,s}$ into five partite sets:$V_1 = \{v\}$,

$V_2 = \{u : u$ is not adjacent to v in first copy of $K_{r,s}\}$,
$V_3 = \{u' : u'$ is not adjacent to v second copy of $K_{r,s}\}$,
$V_4 = \{w : w$ is adjacent to v in first copy of $K_{r,s}\}$ and
$V_5 = \{w' : w'$is adjacent to v insecond copy of $K_{r,s}\}$. These partite sets lead to the quotient matrix $A(G/\pi) = \begin{pmatrix} 0 & 0 & 0 & r & r \\ 0 & 0 & 0 & r & 0 \\ 0 & 0 & 0 & 0 & r \\ 1 & s-1 & 0 & 0 & 0 \\ 1 & 0 & s-1 & 0 & 0 \end{pmatrix}$. UsingTheorem 2.2, the adjacency quotient matrix of complement of $\vee_v[K_{r,s}, K_{r,s}]$is

$$A(G^c/\pi) = J(\pi) - A(G/\pi) - I = \begin{pmatrix} 0 & s-1 & s-1 & 0 & 0 \\ 1 & s-2 & s-1 & 0 & r \\ 1 & s-1 & s-2 & r & 0 \\ 0 & 0 & s-1 & r-1 & r \\ 0 & s-1 & s-1 & r & r-1 \end{pmatrix},$$

where $J(\pi) = \begin{pmatrix} 1 & s-1 & s-1 & r & r \\ 1 & s-1 & s-1 & r & r \\ 1 & s-1 & s-1 & r & r \\ 1 & s-1 & s-1 & r & r \\ 1 & s-1 & s-1 & r & r \end{pmatrix}$.

By Proposition 2.1, we have $\phi(G^c/\pi) = x^5 - (2r + 2s - 6)x^4 - (8s - 2rs + 8r - 14)x^3 - (12s - sr^2 + r^2 - s^2r - 6rs + 13r - 16)x^2 - (8s + r^2s^2 - 3sr^2 + 2r^2 - 2s^2r - 6rs + 10r - 9)x - (2s + 2r^2s^2 - 4sr^2 + 2r^2 - s^2r - 2rs + 3r - 2)$ and by Theorem 2.2 remaining part of the spectrum of complement of $\vee_v[K_{r,s}, K_{r,s}]$is $\{-1^{2r+2s-6}\}$. Now, by using Proposition 2.3, the Seidel quotient matrix of $\vee_v[K_{r,s}, K_{r,s}]$is given by

$$S(G/\pi) = \begin{pmatrix} 0 & s-1 & s-1 & -r & r \\ 1 & s-2 & s-1 & -r & r \\ 1 & s-1 & s-2 & r & -r \\ 0 & -(s-1) & s-1 & r-1 & r \\ 0 & s-1 & -(s-1) & r & r-1 \end{pmatrix}.$$

By Proposition 2.1, we have
$\phi(S(G/\pi)) = x^5 - (2r + 2s - 6)x^4 - (8s + 6r - 14)x^3 - (12s - 8sr^2 + 8r^2 - 8s^2r + 12rs + 2r - 16)x^2 - (8s + 16r^2s^2 - 40sr^2 + 24r^2 - 16s^2r + 24rs - 6r - 9)x - (2s + 16r^2s^2 - 32sr^2 + 16r^2 - 8s^2r + 12rs - 4r - 2)$ andfrom Theorem 2.4 the remaining part of the Seidel spectrum of $\vee_v[K_{r,s}, K_{r,s}]$ is $\{-1^{2r+2s-6}\}$. Hence the result follows.

It is noted that when $r = s$ there is no distinction between the vertices among the two partitions of a copy of $K_{r,s}$hence the result follows by putting $r = s$ in the two cases (i) and (ii).

Theorem 3.5: The adjacency polynomial of complement of $\vee_v[K_{r,s}, K_{r,s}]$ $($2 copies of $K_{r,s})$and the Seidel polynomial of $\vee_v[K_{r,s}, K_{r,s}]$(2 copies of $K_{r,s}$) where $r < s$ are

$$\phi(\vee_v[K_{r,s}, K_{r,s}]:\ x) = \begin{cases} (x+1)^{2r+2s-6}(x^5 - (2r+2s-6)x^4 - (8r-2rs+8s-14)x^3 \\ -(12r - rs^2 + s^2 - r^2s - 6rs + 13s - 16)x^2 \\ -(8r + r^2s^2 - 3rs^2 + 2s^2 - 2r^2s - 6rs + 10s - 9)x \\ -(2r + 2r^2s^2 - 4rs^2 + 2s^2 - r^2s - 2rs + 3s - 2)) \\ if\ v \text{ is selected among the } r \text{ vertices of } K_{r,s}, \\ (x+1)^{2r+2s-6}(x^5 - (2r+2s-6)x^4 - (8s-2rs+8r-14)x^3 \\ -(12s - sr^2 + r^2 - s^2r - 6rs + 13r - 16)x^2 \\ -(8s + r^2s^2 - 3sr^2 + 2r^2 - 2s^2r - 6rs + 10r - 9)x \\ -(2s + 2r^2s^2 - 4sr^2 + 2r^2 - s^2r - 2rs + 3r - 2)), \\ if\ v \text{ is selected among the } s \text{ vertices of } K_{r,s} \end{cases}$$

with $\phi(\vee_v[K_{r,s}, K_{r,s}]:\ x)$=$(x+1)^{2r+2s-6}(x^5 - (4r-6)x^4 - (8r - 2r^2 + 8r - 14)x^3 - (12r - 21r^3 - 5r^2 + 13r - 16)x^2 - (8r + r^4 - 5r^3 - 4r^2 + 10r - 9)x - (2r + 2r^4 - 4r^2 - r^3 + 3r - 2))$ when $r = s$

and

$$\phi(\vee_v[K_{r,s}, K_{r,s}]:\ x) = \begin{cases} (x+1)^{2r+2s-6}(x^5 - (2r+2s-6)x^4 - (8r+6s-14)x^3 \\ -(12r - 8rs^2 + 8s^2 - 8r^2s + 12rs + 2s - 16)x^2 \\ -(8r + 16r^2s^2 - 40rs^2 + 24s^2 - 16r^2s + 24rs - 6s - 9)x \\ -(2r + 16r^2s^2 - 32rs^2 + 16s^2 - 8r^2s + 12rs - 4s - 2)) \\ if\ v \text{ is selected among the r vertices of } K_{r,s}, \\ (x+1)^{2r+2s-6}(x^5 - (2r+2s-6)x^4 - (8s+6r-14)x^3 \\ -(12s - 8sr^2 + 8r^2 - 8s^2r + 12rs + 2r - 16)x^2 \\ -(8s + 16r^2s^2 - 40sr^2 + 24r^2 - 16s^2r + 24rs - 6r - 9)x \\ -(2s + 16r^2s^2 - 32sr^2 + 16r^2 - 8s^2r + 12rs - 4r - 2)), \\ if\ v \text{ is selected among the } s \text{ vertices of } K_{r,s} \end{cases}$$

with $\phi(\vee_v[K_{r,s},K_{r,s}]:\ x)=(x+1)^{2r+2s-6}(x^5-(4r-6)x^4-(14r-14)x^3-(-16r^3+20r^2+14r-16)x^2-(16r^4-56r^3+48r^2+2r-9)x-(16r^4+28r^2-40r^3-2r-2))$ when $r=s$.

Proof: (i) when v is selected among r vertices of $K_{r,s}$, making the vertices of $\vee_v[K_{r,s},K_{r,s}]$ which includes two copies of $K_{r,s}$ into five partite sets: $V_1=\{v\}$,
$V_2=\{u: u \text{ is not adjacent to } v \text{ in first copy of } K_{r,s}\}$,
$V_3=\{u': u' \text{ is not adjacent to } v \text{ second copy of } K_{r,s}\}$,
$V_4=\{w: w \text{ is adjacent to } v \text{ in first copy of } K_{r,s}\}$ and
$V_5=\{w': w' \text{is adjacent to } v \text{ insecond copy of } K_{r,s}\}$. These partite sets lead to the quotient matrix $A(G/\pi)=\begin{pmatrix}0&0&0&s&s\\0&0&0&s&0\\0&0&0&0&s\\1&r-1&0&0&0\\1&0&r-1&0&0\end{pmatrix}$. UsingTheorem 2.2, the adjacency quotient matrix of complement of $\vee_v[K_{r,s},K_{r,s}]$is

$$A\left(\frac{G^c}{\pi}\right)=J(\pi)-A\left(\frac{G}{\pi}\right)-I=\begin{pmatrix}0&r-1&r-1&0&0\\1&r-2&r-1&0&s\\1&r-1&r-2&s&0\\0&0&r-1&s-1&s\\0&r-1&r-1&s&s-1\end{pmatrix},\text{where}$$

$$J(\pi)=\begin{pmatrix}1&r-1&r-1&s&s\\1&r-1&r-1&s&s\\1&r-1&r-1&s&s\\1&r-1&r-1&s&s\\1&r-1&r-1&s&s\end{pmatrix}.$$

By Proposition 2.1, we have
$\phi(G^c/\pi)=x^5-(2r+2s-6)x^4-(8r-2rs+8s-14)x^3-(12r-rs^2+s^2-r^2s-6rs+13s-16)x^2-(8r+r^2s^2-3rs^2+2s^2-2r^2s-6rs+10s-9)x-(2r+2r^2s^2-4rs^2+2s^2-r^2s-2rs+3s-2)$and by Theorem 2.2 remaining part of the spectrum of complement of $\vee_v[K_{r,s},K_{r,s}]$is $\{-1^{2r+2s-6}\}$. Now, by using Proposition 2.3, the Seidel quotient matrix of $\vee_v[K_{r,s},K_{r,s}]$is $S(G/\pi)=$

$$\begin{pmatrix}0&r-1&r-1&-s&s\\1&r-2&r-1&-s&s\\1&r-1&r-2&s&-s\\0&-(r-1)&r-1&s-1&s\\0&r-1&-(r-1)&s&s-1\end{pmatrix}.$$

By Proposition 2.1, we have

$\phi(S(G/\pi)) = x^5 - (2r + 2s - 6)x^4 - (8r + 6s - 14)x^3 - (12r - 8rs^2 + 8s^2 - 8r^2s + 12rs + 2s - 16)x^2 - (8r + r^2s^2 - 3rs^2 + 2s^2 - 2r^2s - 6rs + 10s - 9)x - (2r + 2r^2s^2 - 4rs^2 + 2s^2 - r^2s - 2rs + 3s - 2)$ and from Theorem 2.4 the remaining part of the Seidel spectrum of $\vee_v[K_{r,s}, K_{r,s}]$ is $\{-1^{2r+2s-6}\}$. Hence the result follows.

(ii) when v is selected among s vertices of $K_{r,s}$, making the vertices of $\vee_v[K_{r,s}, K_{r,s}]$ which includes two copies of $K_{r,s}$ into five partite sets: $V_1 = \{v\}$,

$V_2 = \{u: u$ is not adjacent to v in first copy of $K_{r,s}\}$,

$V_3 = \{u': u'$ is not adjacent to v second copy of $K_{r,s}\}$,

$V_4 = \{w: w$ is adjacent to v in first copy of $K_{r,s}\}$ and

$V_5 = \{w': w'$is adjacent to v insecond copy of $K_{r,s}\}$. These partite sets lead to the quotient matrix $A(G/\pi) = \begin{pmatrix} 0 & 0 & 0 & r & r \\ 0 & 0 & 0 & r & 0 \\ 0 & 0 & 0 & 0 & r \\ 1 & s-1 & 0 & 0 & 0 \\ 1 & 0 & s-1 & 0 & 0 \end{pmatrix}$.

Using Theorem 2.2, the adjacency quotient matrix of complement of $\vee_v[K_{r,s}, K_{r,s}]$is

$$A(G^c/\pi) = J(\pi) - A(G/\pi) - I = \begin{pmatrix} 0 & s-1 & s-1 & 0 & 0 \\ 1 & s-2 & s-1 & 0 & r \\ 1 & s-1 & s-2 & r & 0 \\ 0 & 0 & s-1 & r-1 & r \\ 0 & s-1 & s-1 & r & r-1 \end{pmatrix}$$

where $J(\pi) = \begin{pmatrix} 1 & s-1 & s-1 & r & r \\ 1 & s-1 & s-1 & r & r \\ 1 & s-1 & s-1 & r & r \\ 1 & s-1 & s-1 & r & r \\ 1 & s-1 & s-1 & r & r \end{pmatrix}$.

By Proposition 2.1, we have $\phi(G^c/\pi) = x^5 - (2r + 2s - 6)x^4 - (8s - 2rs + 8r - 14)x^3 - (12s - sr^2 + r^2 - s^2r - 6rs + 13r - 16)x^2 - (8s + r^2s^2 - 3sr^2 + 2r^2 - 2s^2r - 6rs + 10r - 9)x - (2s + 2r^2s^2 - 4sr^2 + 2r^2 - s^2r - 2rs + 3r - 2)$ and by Theorem 2.2 remaining part of the spectrum of complement of $\vee_v[K_{r,s}, K_{r,s}]$is $\{-1^{2r+2s-6}\}$. Now, by using Proposition 2.3, the Seidel quotient matrix of $\vee_v[K_{r,s}, K_{r,s}]$is given by

$$S(G/\pi) = \begin{pmatrix} 0 & s-1 & s-1 & -r & r \\ 1 & s-2 & s-1 & -r & r \\ 1 & s-1 & s-2 & r & -r \\ 0 & -(s-1) & s-1 & r-1 & r \\ 0 & s-1 & -(s-1) & r & r-1 \end{pmatrix}.$$

By Proposition 2.1, we have

$\phi(S(G/\pi)) = x^5 - (2r + 2s - 6)x^4 - (8s + 6r - 14)x^3 - (12s - 8sr^2 + 8r^2 - 8s^2r + 12rs + 2r - 16)x^2 - (8s + 16r^2s^2 - 40sr^2 + 24r^2 - 16s^2r + 24rs - 6r - 9)x - (2s + 16r^2s^2 - 32sr^2 + 16r^2 - 8s^2r + 12rs - 4r - 2)$ and from Theorem 2.4 the remaining part of the Seidel spectrum of $\vee_v[K_{r,s}, K_{r,s}]$ is $\{-1^{2r+2s-6}\}$. Hence the result follows.

It is noted that when $r = s$ there is no distinction between the vertices of two partitions of a copy of $K_{r,s}$hence the result follows by putting $r = s$ in the two cases (i) and (ii).

4. Conclusion

In this article, we have studied the characteristic polynomial of two well defined matrices associated to a graph, the adjacency matrix and the Seidel matrix by using the graph operations like splice and link through equitable partition.

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