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ON THE BOUNDS AND ENERGY OF BLOCK ADJACENCY MATRIX FOR SOME CLASS OF GRAPHS

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Abstract

In the present research work, effort has been made to define new energy with respect to block adjacency concept. Hence a new kind of block adjacency matrix BA(G) is introduced.The block adjacency energy of the graph is defined as the sum of the absolute values of the eigenvalues of block adjacency matrix.The results are established on spectra and energy of block adjacency of matrix for some class of graphs. Further we obtained the bounds for eigenvalues and energy for block adjacency energy for the some class of graphs.

Keywords: Block adjacency of matrix, Block adjacency, Block adjacency Spectrum.

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1. Introduction

The genesis of graph energy concept was traced back to chemistry. The famous Huckel Molecular Orbital Theory was proposed by Erich Huckel in 1930. In theoretical chemistry, Huckel theory is used to compute π-electron energy of a conjugated hydrocarbon molecule. This motivates mathematician to define the concept of graph energy. Gutman in 1978[8] introduced the concept of graph energy. Very recently graph energy has become a matter of interest to mathematicians and inspired to carryout research in various innovative concepts of graph energy.

Let G be a simple, finite, undirected graph with n vertices and m edges. Undefined terminologies are referred from [9].

The adjacency matrix[8] of the graph G is the symmetric square matrix denoted by $A(G) = (a_{ij})$ of order n whose (i, j)-entry is defined as

$$A(G) = (a_{ij}) = \begin{cases} 1, & if\ v_i\ and\ v_j\ are\ adjacent\,; \\ 0, & otherwise\,. \end{cases}$$

The eigenvalues $\lambda_1, \lambda_2, \lambda_3, .. \lambda_n$ of $A(G)$, assumed in nonincreasing order are the eigenvalues of the graph G. Since $A(G)$ is a symmetric matrix with zero trace, these eigenvalues are real with sum equal to zero.

The sum of the absolute eigenvalues of graph is called as graph energy $E_A(G)$ [8].

$$E_A(G) = \Sigma_i^n |\lambda_i| \qquad (1)$$

Where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_n$ are the eigenvalues of $A(G)$ matrix.

The collection of these eigenvalues along with their multiplicities is known as the spectrum of a graph G[5].

$$Spec(A)(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \ldots & \lambda_n \\ m(\lambda_1) & m(\lambda_2) & m(\lambda_3) & \ldots & m(\lambda_n) \end{pmatrix} \qquad (2)$$

Inspired by the research work of [1] and [10] on energy, we defined and investigated block adjacency matrix and its properties.

Blockadjacency matrix.LetG be graph with B-blocks,where B $=\{b_1, b_2, b_3, \ldots b_k; k \in N\}$ be the total number of blocks in G and $B \geq 2$. Then the block adjacency matrix $BA(G) = [b_{ij}]$ is defined as

$$BA(G) = [b_{ij}] = \begin{cases} 1, & if\ b_i\ and\ b_j\ are\ adjacent\,; \\ 0, & otherwise\,. \end{cases}$$

The block adjacency matrix $BA(G)$ is a real symmetric matrix. If $\gamma_1, \gamma_2, \gamma_3 \ldots, \gamma_B$ are eigenvalue of $BA(G)$, then they are arranged as $\gamma_1 \geq \gamma_2 \geq \gamma_3 \geq \ldots \geq \gamma_B$. The block adjacency energy of a graph $E_{BA}(G)$ is defined as

$$E_{BA}(G) = \sum_{i=1}^{B} |\gamma_i|$$

and the block adjacency matrix spectrum is defined as,

$$Spec(BA)(G) = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \cdots & \gamma_B \\ m(\gamma_1) & m(\gamma_2) & m(\gamma_3) & \ldots & m(\gamma_B) \end{pmatrix} \quad (3)$$

The **Helm graph** H_t [7], where $t \geq 3$ indicates the number of pendent edges, is the graph obtained from a n-wheel graph W_n by joining a pendent edge at each vertex of the cycle. The maximum number of blocks in H_t are $(t+1)$.

A $\boldsymbol{n}$**-Barbell graph** B_n [2] is formed by joining each end point of bridge by a complete graph k_n. The maximum number of blocks in B_n are 3.

In this paper, the results are established on energy and spectra of block adjacency matrix of some class of graphs and further bounds for eigenvalue and energy of block adjacency energy are also computed.

2. Preliminaries

Theorem 2.1*[3] The Cauchy-Schwarz inequality states that if $(a_1, a_2, a_3, \ldots, a_n)$ and $(b_1, b_2, b_3, \ldots, b_n)$ are real n-vectors then,*

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

Theorem 2.2*[12] Suppose a_i and b_i, $1 \leq i \leq n$ are positive real numbers, then*

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \leq \frac{1}{4}\left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}}\right)^2 \left(\sum_{i=1}^{n} a_i b_i\right)^2$$

Where $M_1 = max_{1\leq i\leq n}(a_i)$; $M_2 = max_{1\leq i\leq n}(b_i)$; $m_1 = min_{1\leq i\leq n}(a_i)$; $m_2 = min_{1\leq i\leq n}(b_i)$

Theorem 2.3*[11] Let a_i and b_i, $1 \leq i \leq n$ are nonnegative real numbers, then*

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \frac{n^2}{4}(M_1 M_2 - m_1 m_2)^2$$

Where M_1M_2 and m_1m_2 are defined similarly to Theorem 2.2.

Theorem 2.4*[4] Suppose a_i and b_i, $1 \leq i \leq n$ are positive real numbers, then*

$$\left| n\sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \right| \leq \mu(n)(A-a)(B-b)$$

Where $a, b, A\ and\ B$ are real constants, that for each i, $1 \leq i \leq n$, $a \leq a_i \leq A$ and $b \leq b_i \leq B$. Further, $\mu(n) = n\lfloor\frac{n}{2}\rfloor\left(1 - \frac{1}{n}\lfloor\frac{n}{2}\rfloor\right)$.

Theorem 2.5 *[6] Let a_i and b_i, $1 \leq i \leq n$ are nonnegative real numbers, then*

$$\sum_{i=1}^{n} b_i^2 + rR\sum_{i=1}^{n} a_i^2 \leq (r+R)\left(\sum_{i=1}^{n} a_i b_i\right)$$

Where r and R are real constants. So that for each i, $1 \leq i \leq n$ holds $ra_i \leq b_i \leq Ra_i$.

3.Results

3.1. Block adjacency Energy and spectrum of some class of graphs

In this section, the results on block adjacency energy and spectrum for some class of graphs are obtained.

Theorem 3.1*If G be a graph with B mutually adjacent blocks, then the block adjacency energy and spetrum of G is*

$$E_{BA}(G) = 2(B-1)$$

$$Spec(BA)(G) = \begin{pmatrix} -1 & (B-1) \\ (B-1) & 1 \end{pmatrix}$$

Proof. Let G be a graph with mutually adjacent blocks and $BA(G)$ be its block adjacency matrix. The eigenvalues -1 and $(B-1)$ of $BA(G)$ occur with multiplicities $(B-1)$ and 1 respectively. Then by equation (2) block adjacency energy of G is

$$E_{BA}(G) = \Sigma_{i=1}^{B}|\lambda_i|$$

$$E_{BA}(G) = |-(B-1)| + |(B-1)|$$

$$= 2(B-1)$$

Also from equation (3) the spectrum of $BA(G)$ is,

$$Spec(BA)(G) = \begin{pmatrix} -1 & -(B-1) \\ (B-1) & 1 \end{pmatrix}$$

Theorem 3.2*If H_t, $t \geq 3$ be a Helm graph with $B \geq 4$ blocks, then the block adjacency energy and spectrum of H_t is*

$$E_{BA}(H_t) = 2\sqrt{B-1}$$

$$Spec(BA)(H_t) = \begin{pmatrix} -\sqrt{B-1} & 0 & \sqrt{B-1} \\ 1 & (B-2) & 1 \end{pmatrix}$$

Proof. Let H_t be a Helm graph and $BA(H_t)$ be its block adjacency matrix. The eigenvalues $-\sqrt{B-1}$, 0, $\sqrt{B-1}$ of $BA(H_t)$ occur with multiplicities 1, $(B-2)$ and 1 times respectively. Then by equation (2), block adjacency energy of H_t is

$$E_{BA}(G) = \Sigma_{i=1}^{B} |\lambda_i|$$

$$E_{BA}(H_t) = \left|-\sqrt{B-1}\right| + 0 + \left|\sqrt{B-1}\right|$$

$$= 2\sqrt{B-1}$$

Also, from equation (3) the spectrum of $BA(H_t)$ is,

$$Spec(BA)(H_t) = \begin{pmatrix} -\sqrt{B-1} & 0 & \sqrt{B-1} \\ 1 & (B-2) & 1 \end{pmatrix}$$

Theorem 3.3*If B_n be a Barbell graph, then the block adjacency energy and spectrum of B_n is*

$$E_{BA}(B_n) = 2\sqrt{B-1}$$

$$Spec(BA)(B_n) = \begin{pmatrix} -\sqrt{B-1} & 0 & \sqrt{B-1} \\ 1 & 1 & 1 \end{pmatrix}$$

Proof. Let B_n be a Barbell graph and $BA(B_n)$ be its block adjacency matrix. The eigenvalues $-\sqrt{B-1}$, 0, $\sqrt{B-1}$ of $BA(B_n)$ occur with multiplicities 1 each. Then by equation (2), block adjacency energy of B_n is

$$E_{BA}(G) = \Sigma_{i=1}^{B}|\lambda_i|$$

$$E_{BA}(B_n) = |-\sqrt{B-1}| + 0 + |\sqrt{B-1}|$$

$$= 2\sqrt{B-1}$$

Also from equation (3), the spectrum of $BA(B_n)$ is,

$$Spec(BA)(B_n) = \begin{pmatrix} -\sqrt{B-1} & 0 & \sqrt{B-1} \\ 1 & 1 & 1 \end{pmatrix}$$

Remark 3.4*The above result (thorem 3.3) holds for all block graphs which are isomorphic to path graph P_3.*

3.2.Bounds for the eigenvalues and energy of block adjacency matrix of Helm and Barbell graphs.

3.2.1. Bounds for the largest eigenvalue of $BA(G)$

The following lemma is used in the proof of theorems.

Lemma A. The eigenvalues of $BA(G)$ satisfy the following results only if $trace[BA(G)] = 0$.

$$(i) \sum_{i=1}^{B-1} \gamma_i = 0$$

$$(ii) \sum_{i=1}^{B-1} \gamma_i^2 = trace(BA(G))^2$$

$$= \sum_{i=1}^{B-1} \sum_{j=1}^{B-1} {b_{ij}}^2$$

$$= \sum_{i=1}^{B-1} \sum_{j=1}^{B-1} 1^2$$

$$= 2(B-1) = 2S$$

where $S = B - 1$

Theorem 3.5*If G be a graph with B blocks, then*

$$\gamma_1 \leq \sqrt{\frac{2S(B-1)}{B}}$$

Proof. Let G be a graph with B blocks. $BA(G)$ be its block adjacency matrix and $\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_B$ are the eigenvalues. where γ_1 is the largest eigenvalue.Using Cauchy-Schwarz inequality theorem 2.1, the bound for γ_1 is computed as

$$\left(\sum_{i=1}^{B} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{B} a_i^2\right)\left(\sum_{i=1}^{B} b_i^2\right)$$

Let $a_i = 1$ and $b_i = \gamma_i, \forall i = 2,3, \ldots, B$ then the inequality becomes,

$$\left(\sum_{i=2}^{B} (1)(\gamma_i)\right)^2 \leq \left(\sum_{i=2}^{B} 1^2\right)\left(\sum_{i=2}^{B} \gamma_i^2\right) \quad (4)$$

From Lemma A(i),

$$\sum_{i=1}^{B} \gamma_i = 0$$

$$\gamma_1 + \sum_{i=2}^{B} \gamma_i = 0$$

$$\left(\sum_{i=2}^{B} \gamma_i\right)^2 = (-\gamma_1)^2 \quad (5)$$

And from Lemma A(ii),

$$\sum_{i=1}^{B} (\gamma_i)^2 = 2S$$

$$(\gamma_1)^2 + \sum_{i=2}^{B} (\gamma_i)^2 = 2S$$

$$\sum_{i=2}^{B} (\gamma_i)^2 = 2S - (\gamma_1)^2 \quad (6)$$

Substituting (5) and (6) in equation (4), we get

$$(-\gamma_1)^2 \leq (B-1)(2S - \gamma_1^2)$$

$$\gamma_1^2 \leq 2S(B-1) - \gamma_1^2(B-1)$$

$$\gamma_1 \leq \sqrt{\frac{2S(B-1)}{B}}$$

Theorem 3.6*If G be a graph with B blocks, then*

$$\sqrt{2S} \leq E_{BA}(G) \leq \sqrt{2BS}$$

Proof. Let G be a graph with B blocks and $BA(G)$ be its block adjacency matrix. $\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_B$ are the eigenvalues of $BA(G)$. From theorem 2.1, Cauchy-Schwarz inequality is

$$\left(\sum_{i=1}^{B} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{B} a_i^2\right)\left(\sum_{i=1}^{B} b_i^2\right)$$

On assuming $a_i = 1$ and $b_i = |\gamma_i|$, $i = 1,2, \ldots, B$, we get the above inequality as

$$\left(\sum_{i=1}^{B} 1 * |\gamma_i|\right)^2 \leq \left(\sum_{i=1}^{B} 1^2\right)\left(\sum_{i=1}^{B} |\gamma_i|^2\right)$$

$$\left(\sum_{i=1}^{B} |\gamma_i|\right)^2 \leq B\left(\sum_{i=1}^{B} |\gamma_i|^2\right)$$

On simplifying and by using Lemma A(ii), we get

$$E_{BA}(G) \leq \sqrt{2BS} \tag{7}$$

Since , $\left(\sum_{i=1}^{B} |\gamma_i|\right)^2 \geq \sum_{i=1}^{B} |\gamma_i|^2$

By using Lemma A(ii), we get

$$E_{BA}(G) \geq \sqrt{2S} \tag{8}$$

From equation (7) and (8), we get

$$\sqrt{2S} \leq E_{BA}(G) \leq \sqrt{2BS}$$

3.2.2.Lower bounds for the block adjacency energy $\boldsymbol{E_{BA}(G)}$

Theorem 3.7*If G be a graph with B blocks, then*

$$E_{BA}(G) \geq \frac{2\sqrt{2BS|\gamma_1||\gamma_B|}}{|\gamma_1| + |\gamma_B|}$$

Proof. Let G be a graph with B blocks and $|\gamma_1| \geq |\gamma_2| \geq |\gamma_3| \geq \ldots \geq |\gamma_B|$ are the eigenvalues of $BA(G)$. The maximum and minimum eigenvalues of $BA(G)$ are $|\gamma_1|$ and $|\gamma_B|$ respectively.

From theorem 2.2, we have

$$\sum_{i=1}^{B} a_i^2 \sum_{i=1}^{B} b_i^2 \leq \frac{1}{4}\left(\sqrt{\frac{M_1M_2}{m_1m_2}} + \sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2 \left(\sum_{i=1}^{B} a_ib_i\right)^2$$

On assuming $a_i = 1$, $b_i = |\gamma_i|$, $M_1M_2 = |\gamma_1|$ and $m_1m_2 = |\gamma_B|$, we get

$$\sum_{i=1}^{B} 1^2 \sum_{i=1}^{B} |\gamma_i|^2 \leq \frac{1}{4}\left(\sqrt{\frac{|\gamma_1|}{|\gamma_B|}} + \sqrt{\frac{|\gamma_B|}{|\gamma_1|}}\right)^2 \left(\sum_{i=1}^{B} (1)(|\gamma_i|)\right)^2 \qquad (9)$$

By using Lemma A(ii), simplifying equation (9) we get

$$B2S \leq \frac{1}{4}\left[\frac{(|\gamma_1| + |\gamma_B|)^2}{|\gamma_1||\gamma_B|}\right](E_{BA}(G))^2$$

$$(E_{BA}(G))^2 \geq \frac{8BS|\gamma_1||\gamma_B|}{(|\gamma_1| + |\gamma_B|)^2}$$

$$E_{BA}(G) \geq \frac{2\sqrt{2BS|\gamma_1||\gamma_B|}}{|\gamma_1| + |\gamma_B|}$$

Theorem 3.8*If G be a graph with B blocks, then*

$$E_{BA}(G) \geq \sqrt{2BS - \frac{B^2}{4}(|\gamma_1| - |\gamma_B|)^2}$$

Proof. Let G be a graph with blocks B and $|\gamma_1| \geq |\gamma_2| \geq |\gamma_3| \ldots \geq |\gamma_B|$ are the eigen values of $BA(G)$. The maximum and minimum eigenvalues of $BA(G)$ are $|\gamma_1|$ and $|\gamma_B|$ respectively.

From theorem, 2.3 we have the inequality,

$$\sum_{i=1}^{B} a_i^2 \sum_{i=1}^{B} b_i^2 - \left(\sum_{i=1}^{B} a_ib_i\right)^2 \leq \frac{B^2}{4}(M_1M_2 - m_1m_2)^2$$

On assuming $a_i = 1$, $b_i = |\gamma_i|$, $M_1M_2 = |\gamma_1|$ and $m_1m_2 = |\gamma_B|$, we get the above above inequality as

$$\sum_{i=1}^{B} 1^2 \sum_{i=1}^{B} |\gamma_i|^2 - \left(\sum_{i=1}^{B} 1 * |\gamma_i|\right)^2 \leq \frac{B^2}{4}(|\gamma_1| - |\gamma_B|)^2$$

From Lemma A(ii), we get

$$B2S-(E_{BA}(G))^2 \leq \frac{B^2}{4}(|\gamma_1|-|\gamma_B|)^2$$

$$(E_{BA}(G)) \geq \sqrt{2BS-\frac{B^2}{4}(|\gamma_1|-|\gamma_B|)^2}$$

Theorem 3.9*If G be a graph with B blocks, then*

$$E_{BA}(G) \geq \sqrt{2BS-\mu(B)(|\gamma_1|-|\gamma_B|)^2}$$

Proof. Let G be a graph with blocks B and $|\gamma_1| \geq |\gamma_2| \geq |\gamma_3| \geq \ldots \geq |\gamma_B|$ be the eigenvalues of $BA(G)$. The maximum and minimum eigen values of $BA(G)$ are $|\gamma_1|$ and $|\gamma_B|$ respectively.

From theorem 2.4, we have

$$\left|B\sum_{i=1}^{B} a_i b_i - \sum_{i=1}^{B} a_i \sum_{i=1}^{B} b_i\right| \leq \mu(B)(P-a)(Q-b)$$

On assuming $a_i = b_i = |\gamma_i|$, $P = Q = |\gamma_1|$ and $a = b = |\gamma_B|$, we get the above inequality as

$$\left|B\sum_{i=1}^{B} |\gamma_i|^2 - \left(\sum_{i=1}^{B} |\gamma_i|\right)^2\right| \leq \mu(B)(|\gamma_1|-|\gamma_B|)(|\gamma_1|-|\gamma_B|)$$

From Lemma A(ii), we get

$$|B2S-(E_{BA}(G))^2| \leq \mu(B)(|\gamma_1|-|\gamma_B|)^2$$

$$E_{BA}(G) \geq \sqrt{2BS-\mu(B)(|\gamma_1|-|\gamma_B|)^2}$$

Theorem 3.10*If G be a graph with B blocks, then*

$$E_{BA}(G) \geq \frac{2S+B|\gamma_1||\gamma_B|}{|\gamma_1|+|\gamma_B|}$$

Proof. Let G be a graph with B blocks and $|\gamma_1| \geq |\gamma_2| \geq |\gamma_3| \geq \ldots \geq |\gamma_B|$ are the eigenvalues of $BA(G)$. The maximum and minimum eigenvalues of $BA(G)$ are $|\gamma_1|$ and $|\gamma_B|$ respectively.

From theorem 2.5, we have

$$\sum_{i=1}^{B} b_i^2 + rR\sum_{i=1}^{B} a_i^2 \leq (r+R)\left(\sum_{i=1}^{B} a_i b_i\right)$$

On assuming $b_i = |\gamma_i|$, $a_i = 1$, $r = |\gamma_B|$ and $R = |\gamma_1|$, then the above inequality becomes

$$\sum_{i=1}^{B} |\gamma_i|^2 + |\gamma_B||\gamma_1|\sum_{i=1}^{B} 1^2 \leq (|\gamma_B| + |\gamma_1|)\left(\sum_{i=1}^{B} 1 * |\gamma_i|\right)$$

From Lemma A(ii), we get

$$2S + (|\gamma_B||\gamma_1|)(B) \leq (|\gamma_B| + |\gamma_1|)E_{BA}(G)$$

$$E_{BA}(G) \geq \frac{2S + B|\gamma_1||\gamma_B|}{|\gamma_1| + |\gamma_B|}$$

3.3 Bounds for the eigenvalues and energy of block adjacency matrix of graphs with mutually adjacent blocks

3.3.1 Bounds for the largest eigenvalue of $BA(G)$

The following lemma is used in the proofs of theorems.

Lemma B. The eigenvalues of $BA(G)$ satisfy the following results only if $trace[BA(G)] = 0$.

$(i) \sum_{i=1}^{B} \gamma_i = 0$

$(ii) \sum_{i=1}^{B} \gamma_i^2 = trace(BA(G))^2$

$= (B-1)\sum_{i=1}^{B} 1(Since,\ diagonal\ elements\ (B-1)\ occurs\ B\ times)$

$= B(B-1) = D$

where $D = B - 1$

Theorem 3.11*If G be a graph with B blocks, then*

$$\gamma_1 \leq \sqrt{\frac{D(B-1)}{B}}$$

Proof. Let G be a graph G with B blocks and $\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_B$ are the eigenvalues of $BA(G)$. The largest eigenvalue is γ_1 and its bound is computed using theorem 2.1

$$\left(\sum_{i=1}^{B} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{B} a_i^2\right)\left(\sum_{i=1}^{B} b_i^2\right)$$

Let $a_i = 1$ and $b_i = \gamma_i, \forall i = 2,3, \ldots, B$, then the inequality becomes,

$$\left(\sum_{i=2}^{B} (1)(\gamma_i)\right)^2 \leq \left(\sum_{i=2}^{B} 1^2\right)\left(\sum_{i=2}^{B} \gamma_i^2\right) \quad (10)$$

From Lemma B(i),

$$\sum_{i=1}^{B} \gamma_i = 0$$

$$\gamma_1 + \sum_{i=2}^{B} \gamma_i = 0$$

$$\left(\sum_{i=2}^{B} \gamma_i\right)^2 = (-\gamma_1)^2 \quad (11)$$

And from Lemma B(ii),

$$\sum_{i=1}^{B} (\gamma_i)^2 = D$$

$$(\gamma_1)^2 + \sum_{i=2}^{B} (\gamma_i)^2 = D$$

$$\sum_{i=2}^{B} (\gamma_i)^2 = D - (\gamma_1)^2 \quad (12)$$

Substituting (11) and (12) in (10), we get

$$(-\gamma_1)^2 \leq (B-1)(D - \gamma_1^2)$$

$$\gamma_1^2 \leq D(B-1) - \gamma_1^2(B-1)$$

$$\gamma_1 \leq \sqrt{\frac{D(B-1)}{B}}$$

Theorem 3.12*If G be a graph with B blocks, then*

$$\sqrt{D} \leq E_{BA}(G) \leq \sqrt{BD}$$

Proof. Let G be a graph with B blocks and $\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_B$ are the eigenvalues $BA(G)$.

From theorem 2.1, we have

$$\left(\sum_{i=1}^{B} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{B} a_i^2\right)\left(\sum_{i=1}^{B} b_i^2\right)$$

On assuming $a_i = 1$ and $b_i = |\gamma_i|$, $i = 1,2,\ldots,B$, the above inequality becomes

$$\left(\sum_{i=1}^{B} 1 * |\gamma_i|\right)^2 \leq \left(\sum_{i=1}^{B} 1^2\right)\left(\sum_{i=1}^{B} |\gamma_i|^2\right)$$

$$\left(\sum_{i=1}^{B} |\gamma_i|\right)^2 \leq B\left(\sum_{i=1}^{B} |\gamma_i|^2\right)$$

On simplifying and by using Lemma B(ii), we get

$$E_{BA}(G) \leq \sqrt{BD} \tag{13}$$

Since, $\left(\sum_{i=1}^{B} |\gamma_i|\right)^2 \geq \sum_{i=1}^{B} |\gamma_i|^2$

By using Lemma B(ii), we get

$$E_{BA}(G) \geq \sqrt{D} \tag{14}$$

From equation (13) and (14), we get

$$\sqrt{D} \leq E_{BA}(G) \leq \sqrt{BD}$$

3.3.2Lower bounds for the block adjacency energy $\boldsymbol{E_{BA}(G)}$

Theorem 3.13*If G be a graph with B blocks, then*

$$E_{BA}(G) \geq \frac{2\sqrt{BD|\gamma_1||\gamma_B|}}{|\gamma_1| + |\gamma_B|}.$$

Proof. Let G be a graph with B blocks and $|\gamma_1| \geq |\gamma_2| \geq |\gamma_3| \geq \ldots \geq |\gamma_B|$ are the eigenvalues of $BA(G)$. The maximum and minimum eigenvalues of $BA(G)$ are $|\gamma_1|$ and $|\gamma_B|$ are respectively.

From theorem 2.2, we have

$$\sum_{i=1}^{B} a_i^2 \sum_{i=1}^{B} b_i^2 \leq \frac{1}{4}\left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}}\right)^2 \left(\sum_{i=1}^{B} a_i b_i\right)^2$$

On assuming $a_i = 1$, $b_i = |\gamma_i|$, $M_1M_2 = |\gamma_1|$ and $m_1m_2 = |\gamma_B|$ we get

$$\sum_{i=1}^{B} 1^2 \sum_{i=1}^{B} |\gamma_i|^2 \le \frac{1}{4}\left(\sqrt{\frac{|\gamma_1|}{|\gamma_B|}} + \sqrt{\frac{|\gamma_B|}{|\gamma_1|}}\right)^2 \left(\sum_{i=1}^{B} (1)(|\gamma_i|)\right)^2$$

From Lemma B(ii), we get

$$BD \le \frac{1}{4}\left[\frac{(|\gamma_1| + |\gamma_B|)^2}{|\gamma_1||\gamma_B|}\right](E_{BA}(G))^2$$

$$(E_{BA}(G))^2 \ge \frac{4BD|\gamma_1||\gamma_B|}{(|\gamma_1| + |\gamma_B|)^2}$$

$$E_{BA}(G) \ge \frac{2\sqrt{DB|\gamma_1||\gamma_B|}}{|\gamma_1| + |\gamma_B|}$$

Theorem 3.14*If G be a graph with B blocks, then*

$$E_{BA}(G) \ge \sqrt{BD - \frac{B^2}{4}(|\gamma_1| - |\gamma_B|)^2}$$

Proof. Let G be a graph with blocks B and $|\gamma_1| \ge |\gamma_2| \ge |\gamma_3| \ldots \ge |\gamma_B|$ are the eigen values of $BA(G)$. The maximum and minimum eigenvalues of $BA(G)$ are $|\gamma_1|$ and $|\gamma_B|$ are respectively.

From theorem 2.3 we have,

$$\sum_{i=1}^{B} a_i^2 \sum_{i=1}^{B} b_i^2 - \left(\sum_{i=1}^{B} a_i b_i\right)^2 \le \frac{B^2}{4}(M_1M_2 - m_1m_2)^2$$

On assuming $a_i = 1$, $b_i = |\gamma_i|$, $M_1M_2 = |\gamma_1|$ and $m_1m_2 = |\gamma_n|$, the above inequality becomes

$$\sum_{i=1}^{B} 1^2 \sum_{i=1}^{B} |\gamma_i|^2 - \left(\sum_{i=1}^{B} 1 * |\gamma_i|\right)^2 \le \frac{B^2}{4}(|\gamma_1| - |\gamma_B|)^2$$

From Lemma B(ii) and definition of block adjacency energy, we get

$$BD - (E_{BA}(G))^2 \le \frac{B^2}{4}(|\gamma_1| - |\gamma_n|)^2$$

$$(E_{BA}(G)) \geq \sqrt{BD - \frac{B^2}{4}(|\gamma_1| - |\gamma_B|)^2}$$

Theorem 3.15*If G be a graph with B blocks, then*

$$E_{BA}(G) \geq \sqrt{BD - \mu(B)(|\gamma_1| - |\gamma_B|)^2}$$

Proof. Let G be a graph with blocks B and $|\gamma_1| \geq |\gamma_2| \geq |\gamma_3| \geq \ldots \geq |\gamma_B|$ are the eigenvalues of $BA(G)$. The maximum and minimum eigen values of $BA(G)$ are $|\gamma_1|$ and $|\gamma_B|$ respectively.

From theorem 2.4, we have

$$\left|B \sum_{i=1}^{B} a_i b_i - \sum_{i=1}^{B} a_i \sum_{i=1}^{B} b_i\right| \leq \mu(B)(P-a)(Q-b)$$

On assuming $a_i = b_i = |\gamma_i|$, $P = Q = |\gamma_1|$ and $a = b = |\gamma_B|$, the above inequality becomes

$$\left|B \sum_{i=1}^{B} |\gamma_i|^2 - \left(\sum_{i=1}^{B} |\gamma_i|\right)^2\right| \leq \mu(B)(|\gamma_1| - |\gamma_B|)(|\gamma_1| - |\gamma_B|)$$

From Lemma B(ii), we get

$$|BD - (E_{BA}(G))^2| \leq \mu(B)(|\gamma_1| - |\gamma_B|)^2$$

$$E_{BA}(G) \geq \sqrt{BD - \mu(B)(|\gamma_1| - |\gamma_B|)^2}$$

Theorem 3.16*If G be a graph with B blocks, then*

$$E_{BA}(G) \geq \frac{D + B|\gamma_1||\gamma_B|}{|\gamma_1| + |\gamma_B|}$$

Proof. Let a graph G with B blocks and $|\gamma_1| \geq |\gamma_2| \geq |\gamma_3| \geq \ldots \geq |\gamma_B|$ are the eigenvalues of $BA(G)$. The maximum and minimum eigenvalues of $BA(G)$ are $|\gamma_1|$ and $|\gamma_B|$ respectively.

From the theorem 2.5, we have

$$\sum_{i=1}^{B} b_i^2 + rR \sum_{i=1}^{B} a_i^2 \leq (r+R)\left(\sum_{i=1}^{B} a_i b_i\right)$$

On assuming $b_i = |\gamma_i|$, $a_i = 1$, $r = |\gamma_B|$ and $R = |\gamma_1|$, the inequality becomes

$$\sum_{i=1}^{B} |\gamma_i|^2 + |\gamma_B||\gamma_1| \sum_{i=1}^{B} 1^2 \leq (|\gamma_B| + |\gamma_1|)\left(\sum_{i=1}^{B} 1 * |\gamma_i|\right)$$

From Lemma B(ii), we get

$$D + (|\gamma_B||\gamma_1|)(B) \leq (|\gamma_B| + |\gamma_1|)E_{BA}(G)$$

$$E_{BA}(G) \geq \frac{D + B|\gamma_1||\gamma_B|}{|\gamma_1| + |\gamma_B|}$$

4. Conclusion

In this paper a novel approach is made to introduce and define block adjacency matrix. The results exhibit the insight into the establishment of energy and spectra of Helm graph, Barbell graph and graph with mutually adjacent blocks. Hence it is inferred that the block adjacency energy of graph with mutual adjacent blocks is same as energy of complete graph K_n and is the highest one among all the graphs with blocks. The result obtained for Barbell graph holds for all the graphs with blocks which are isomorphic to path graph P_3. In continuation, the bounds for eigenvalues and energy for block adjacency matrix are obtained for the same class of graphs.

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