Mathematical Forum ISSN: 0972-9852
Vol. 29, 2022

On the graph of the nilpotent elements of a module

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Received on: 19/10/2020
Accepted on: 09/08/2021

Abstract

In this paper, we introduce the notion of the graph of the nilpotent elements of a module denoted by $\Gamma(N(M))$, $N(M))$ being the set of all nilpotent elements of the module M. Among various interesting properties of this graph, we particularly note that $\Gamma(N(M))$ is either connected with $diam(\Gamma(N(M))) = 4$ or else totally disconnected. We also establish that the notion of bipartite graph and star graph coincide in case of $\Gamma(N(M))$. We investigate conditions under which the collection of all nilpotent elements of a module forms a submodule under graph theoretic settings.

Keywords: nilpotent element, module, submodule.

2010 AMS classification: 16D10, 05

1. Introduction

The study of graphs associated to algebraic structures is one of the most active research areas. Many fundamental papers on graphs defined on the basis of various properties of rings and modules have appeared recently, for instance see, [2],[3],[4],[6], [8] and [9] etc. Further, the notion of nilpotent elements of a ring plays a key role in ring theory in the sense that it provides insights towards the structure theory of rings, viz. Artinian semisimple rings. For instance, see [1], [5]. Recently, Nikmehr et. al. [11] studied graph theoretic aspects of nilpotent elements of a ring.

Throughout this paper, we assume that R is a ring with unity 1 and M is assumed to be a unital left R-module. In [12], the notion of nilpotent element of a module is introduced. A non-zero element m of an R-module M *is* said to be nilpotent if there exists some $a \in R$ such that $a^k m = 0$ but $am \neq 0$ for some $k \in \mathbb{N}$.

In the aforesaid paper, the fact that whether the collection of nilpotent elements of a module M forms a submodule of M is treated as an open problem. In this work, we propose to study this problem.

Throughout this discussion, all graphs are undirected. By $G = (V(G), E(G))$ or simply by G, we mean an undirected graph with the vertex set $V(G)$ and edge set $E(G)$ unless otherwise mentioned. A subgraph of G is a graph having all of its vertices and edges in G. A vertex deleted subgraph of a graph G is the subgraph obtained by the removal of a set of vertices $S \subseteq V(G)$ and all the edges having a vertex belonging to S as end vertex from G. The degree of a vertex v in a graph G is the number of edges incident with v. The degree of a vertex v is denoted by $deg(v)$. The vertex v is *isolated* if $deg(v) = 0$. A walk in G is an alternating sequence of vertices and edges, $v_0 x_1 v_1 ... x_n v_n$ in which each edge x_i is $v_{i-1}v_i$. The length of such a walk is n, the number of occurrences of edges in it. A closed walk has the same first and last vertices. A path is a walk in which all vertices are distinct; a cycle or circuit is a closed walk with all points distinct (except the first and last). For example, a triangle is a cycle of length 3. G is connected if there is a path between every two distinct vertices. A graph which is not connected is called a disconnected graph. A totally disconnected graph does contain no edges. For distinct vertices x and y of G, the distance $d(x, y)$ is the length of the shortest path from x to $y$if the vertices $x, y \in G$ are connected and if there is no such path we define $d(x, y) = \infty$. Then, the diameter of the graph G is

$$diam(G) = sup\{d(x, y) | x, y \in G\}.$$

If in a graph any two vertices are adjacent, it is called a complete graph, dented by K_α where α is the number of vertices of the graph. The length of the shortest cycle in G is called *girth* of G and is denoted by $g(G)$. If G has no cycle then we define $g(G) = \infty$. The graph G is said to be a bipartite graph or bi-graph if its vertex set V can be partitioned into two disjoint subsets V_1 and V_2 with every edge of G joins a vertex of V_1 and a vertex of V_2. If $|V_1| = \alpha$ and $|V_2| = \beta$ and every vertex of V_1 is adjacent to every vertex of V_2, then G is called a complete bipartite graph, denoted by $K_{\alpha,\beta}$. A cut vertex of a graph G is a vertex of G whose removal increases the number of components of G. The chromatic number of a graph G, denoted by $\chi(G)$, is defined to be the minimum number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors.

Any undefined terminology can be obtained in [7] and [10].

2. The graph of nilpotent elements

Let M be an R-module. We define the graph $\Gamma(N(M))$ of the nilpotent elements of M by considering the vertex set to be collection of all nilpotent elements of M, denoted by $N(M)$ and two distinct points $x, y \in N(M)$ are adjacent if and only if $\exists$ some non-zero $r \in R$ such that $r(x + y) \in N(M)$.

Example 1:

(i) for $\mathbb{Z}_n$ where n is a prime or a square-free natural number considered as a module over itself, we have $\Gamma(N(\mathbb{Z}_n))$ is null graph with one vertex.

(ii) for $\mathbb{Z}_4$ considered as a module over itself, we have $N(\mathbb{Z}_4) = \{0, 1, 3\}$ and $\Gamma(N(\mathbb{Z}_4))=(V(G), E(G))$ where $V(G) = N(\mathbb{Z}_4)$ and $E(G) = \{(0, 1), (0, 3), (1, 3)\}$. It is easy to observe that $\Gamma(N(\mathbb{Z}_4))$ is K_3.

Theorem 2.1: $\Gamma(N(M))$ *connected if and only if* $N(M) \neq \{0\}$. *Moreover, if* $\Gamma(N(M))$ *is connected, then* $diam(\Gamma(N(M))) = 4$.

Proof. If $\Gamma(N(M))$ is connected then there is a path between any pair of distinct vertices $x, y \in N(M)$. Consequently, there is at least one non-trivial element in $N(M)$, that is, $N(M) \neq \{0\}$. Conversely, let $N(M) \neq \{0\}$ and let $x, y \in N(M)$ be any two distinct elements. We note that $1.(x + 0) = 1.x = x \in N(M)$ and $1.(y + 0) = 1.y = y \in N(M)$. Thus, $x - 0 - y$ is a path in between $x, y \in N(M)$. Thus, $\Gamma(N(M))$ is connected.

Now, if $\Gamma(N(M))$ is connected, then we have the following cases for any two vertices $x, y \in N(M)$.

Case 1: If x, y are adjacent in $\Gamma(N(M))$, then we have a path $x - y$ of length 1 between x and y, which gives $d(x,y) = 1$.

Case 2: If x, y are not adjacent in $\Gamma(N(M))$, then as we always have a path $x - 0 - y$ of length 2 between x and y, so we have $d(x,y) = 2$.

Case 3: If x,y are not adjacent in $\Gamma(N(M))$, then as for $x \in N(M)$ gives $-x \in N(M)$, we can obtain a path $x - (-x) - 0 - (-y) - y$ of length 4 between x and y which yields $d(x,y) = 4$. Therefore, for any two vertices x and y in $\Gamma(N(M))$, we have $d(x, y) = \{1, 2, 4\}$. Hence, $diam(\Gamma(N(M))) = 4$.

Theorem 2.2: $\Gamma(N(M))$ *is totally disconnected if and only if* $N(M) = \{0\}$.

Proof. Let $\Gamma(N(M))$ be totally disconnected. Then any pair of distinct vertices $x, y \in N(M)$ are not adjacent in $\Gamma(N(M))$. In particular, $0 \in N(M)$ is not adjacent to any non-zero $x \in N(M)$. Thus, $r(x + 0) = rx \notin N(M)$ for any $r \in R$. Thus, in particular, $1.x = x \notin N(M)$ for $1 \in R$. Consequently, $N(M) = \{0\}$. Conversely, let us assume that $N(M) = \{0\}$. Thus, $\Gamma(N(M))$ has the isolated vertex and is a null graph and hence totally disconnected.

Corollary 2.3: $\Gamma(N(M))$ *is either connected or else totally disconnected.*

Proof. The result is clear from Theorem 2.1 and Theorem 2.2.

Corollary 2.4: *If $\Gamma(N(M))$ is disconnected then the number of components of $\Gamma(N(M))$ equals the size of $N(M)$.*

Proof. The result is obvious from corollary 2.3.

We know that if a graph is a star graph, then it is obviously a bipartite graph. Also a star graph has only one cut vertex. But the converse of none of the statements follows in general. There exists a plethora of examples in literature in this regard. However this is not the case for $\Gamma(N(M))$ as the following result shows:

Theorem 2.5: *The following statements are equivalent for $\Gamma(N(M))$.*

(i) $\Gamma(N(M))$ is a bipartite graph.

(ii) $\Gamma(N(M))$ is a star graph.

(iii) 0 is the only cut vertex of $\Gamma(N(M))$.

Proof.

$(i) \Rightarrow (ii)$ Let $\Gamma(N(M))$ be a bipartite graph. Then, there exists two partitions U, V of $N(M)$ such that every $x \in U$ is adjacent to some $y \in V$. Since $0 \in N(M)$ hence 0 belongs to one of U and V. We note that any one of U or V containing 0 cannot contain any other non-trivial vertex of $\Gamma(N(M))$ as for any non-zero $x \in U$ or V, $1.(x + 0) = x$ yields 0 is adjacent to x, a contradiction to the fact that $\Gamma(N(M))$ is bipartite. Thus the only possibility is that one of U or V contains only 0 and the other one contains all other non-trivial vertices of $\Gamma(N(M))$. Consequently, $\Gamma(N(M))$ is a star graph.

$(ii) \Rightarrow (i)$ is obvious as every star graph is a bipartite graph.
$(ii) \Rightarrow (iii)$ follows trivially.

$(iii) \Rightarrow (ii)$: Let 0 be the only cut vertex of $\Gamma(N(M))$. Then removal of 0 will make $\Gamma(N(M))$ adisconnected graph. In that case, in light of corollary 2.3, we may conclude that $\Gamma(N(\mathrm{M}))$ is totally disconnected. This fact forces $\Gamma(N(M))$ to be a star graph.

Corollary 2.6: *If 0 is the only cut vertex of $\Gamma(N(M))$ then $\chi(\Gamma(N(M))) = 2$.*

Proof. If 0 is the only cut vertex of $\Gamma(N(M))$ then by theorem 2.5, $\Gamma(N(M))$ is bipartiteand so $\chi(\Gamma(N(M))) = 2$.

Theorem 2.7: *If $\Gamma(N(M))$ has a cycle Cn, $n \geq 4$ then $\Gamma(N(M))$ must contain a triangle.*

Proof. First, let us assume that $C_n : v_0 - v_1 - ... - v_n$ be a cycle of length $n \geq 4$ in $\Gamma(N(M))$. If 0 is in C_n, then for any two adjacent points v_i, v_j in C_n we obtain a triangle $0-v_i-v_j-0$ as 0 is adjacent to every vertex in $\Gamma(N(M))$. If 0 is a vertex lying outside C_n, then also we can find a triangle $0-v_i-v_j-0$ in $\Gamma(N(M))$for any two adjacent points v_i, v_j in C_n because of the same reason. Hence, $\Gamma(N(M))$ must contain a triangle.

Theorem 2.8: *If $\Gamma(N(M))$ is a tree or 0 is the only cut vertex of $\Gamma(N(M))$, then $g(\Gamma(N(M))) = \infty$ and in other cases, $g(\Gamma(N(M))) = 3$. Thus, $g(\Gamma(N(M))) \in \{3, \infty\}$.*

Proof. If $\Gamma(N(M))$ is a tree then obviously $g(\Gamma(N(M))) = \infty$. Also if 0 is the only cut vertex of $\Gamma(N(M))$, then by Theorem 2.5, we have $\Gamma(N(M))$ is a star graph or bipartite graph which yields $g(\Gamma(N(M))) = \infty$.
In other cases, $\Gamma(N(M))$ has a cycle. Then by Theorem 2.7, $\Gamma(N(M))$ must contain a triangle and hence $g(\Gamma(N(M))) = 3$.

3. When $N(M)$ is a submodule?

As noted in the introduction, $N(M)$ does not in general form a submodule of M. In this section, we investigate conditions under which the same follows.

Theorem 3.1: *Consider the following statements:*

(i) $N(M)$ is a submodule of M.
(ii) $\Gamma(N(M))$ is a complete graph.
(iii) Every vertex is adjacent to 0 in $\Gamma(N(M))$.
Then, $(i) \Rightarrow (ii) \Rightarrow (iii)$

Proof. $(i) \Rightarrow (ii)$

Let $\Gamma(N(M))$ is a submodule of M. This implies that for any $x, y \in N(M)$, $x + y \in N(M)$. This in turn gives $1.(x+y) = x+y \in N(M)$. Consequently, any two vertices in $N(M)$ areadjacent in $\Gamma(N(M))$. Thus $\Gamma(N(M))$ is a complete graph.

$(ii) \Rightarrow (iii)$ clear.

Remark: The reverse implications do not hold in general. For instance, if we consider $\mathbb{Z}_4$ as a module over itself, then it can be seen from Example 1 that $\Gamma(N(\mathbb{Z}_4))$ is a complete graph, but $N(\mathbb{Z}_4) = \{0, 1, 3\}$ is not a submodule of $\mathbb{Z}_4$. Thus, (ii) does not imply (i).
To see (iii) does not imply (ii) in general, we consider the ring

$R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{Z}_2 \right\}$

and consider the subring S of R as

$$S = \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} \middle| a \in \mathbb{Z}_2 \right\}$$

We further set $M = S^R$. Then $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are nilpotents in X and for $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in M$, it can be seen that $A^2X = 0$ and $A^2Y = 0$ but $AX \neq 0$ and $AY \neq 0$ where 0 is the zero matrix of M.

Thus $X, Y \in N(M)$. Again for $X + Y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ we see that $Z = A(X + Y) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ which is not nilpotent as $AZ = 0$ in M. Thus $X + Y \notin N(M)$. Consequently, X and Y are not adjacent in $\Gamma\big(\mathrm{N}(M)\big)$ although X and Y are adjacent to 0 in $\Gamma\big(\mathrm{N}(M)\big)$.

Theorem 3.2: *For the ring R, let* $2 \in Z(R)$ *where* $Z(R)$ *denotes the set of all zero- divisors of R. Let* $\Gamma(N(M))$ *be a complete graph. Then* $N(M)$ *is a submodule of M provided the collection* $I_x = \{r \in R : rx \in N(M)\}$ *forms a left ideal of R for each* $x \in N(M)$.

Proof. Let $\Gamma(N(M))$ be complete. Then any two distinct vertices in $N(M)$ will be adjacentin $\Gamma(N(M))$. Also, we note that $x + x = (1+1).x = 2.x = 0.x = 0 \in N(M)$. Thus, $N(M)$ isclosed under addition. In order to show tha t $N(M)$ is a submodule of M, we need to show that $N(M)$ is closed under scalar multiplication, that is, the collection $I_X = \{r \in R : r$ forms a left ideal of R for every $x \in N(M)$. Clearly, $1 \in I_x$. Thus if I_x forms a left ideal for each $x \in N(M)$, then by the fact that $1 \in I_x$ for each $x \in N(M)$, one gets $I_X = R$. Consequently, $N(M)$ is closed under scalar multiplication and hence $N(M)$ is a submodule of M.

Theorem 3.3: *The graph* $\Gamma(N(M))$ *is complete if and only if* 0 *is adjacent to every non-trivial vertex of* $N(M)$ *and the vertex deleted subgraph* $\Gamma(N(M)) - 0$ *is complete.*

Proof. Let $\Gamma(N(M)) - 0$ be complete and 0 is adjacent to every vertex in $\Gamma(N(M))$. Then it follows that adjoining 0 to $\Gamma(N(M)) - 0$ makes the graph complete, that is, $\Gamma(N(M)$ iscomplete. Conversely, let $\Gamma(N(M))$ be complete. Then the removal of any vertex from it makes the resultant vertex deleted subgraph complete. Thus, it follows that $\Gamma(N(M)) - 0$ is complete and 0 is adjacent to every vertex in $\Gamma(N(M))$, by Theorem 3.1.

4. Conclusion

In this paper, we have studied a graph structure $\Gamma(N(M))$ of the nilpotent elements ofa module M by considering the set of all nilpotent elements of M, namely, $N(M))$ as vertices. We have proved that $\Gamma(N(M))$ is either connected with $diam(\Gamma(N(M))) = 4$ or else totallydisconnected. Further, we have established the fact that $\Gamma(N(M))$ is bipartite if and onlyif 0 is the only cut vertex of $\Gamma(N(M))$. We have also seen that if $N(M)$ is a submodule of M, then $\Gamma(N(M))$ is a complete graph. Finally we have provided a necessary and sufficient condition for $\Gamma(N(M))$ to be complete.

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