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ON CERTAIN CLASSES OF CONCIRCULARLY FLAT *SP*-KENMOTSU MANIFOLDS ADMITTING QUARTER-SYMMETRIC METRIC CONNECTION

K. L. SAI PRASAD[1] and S. SUNITHA DEVI[2]

[1]*Department of Mathematics, Gayatri Vidya Parishad College of Engineering for Women, Visakhapatnam-530048, India*

[2]*Department of Mathematics, Vignan's Institute of Information Technology, Visakhapatnam-530049, India*

Email : [1]klsprasad@yahoo.com, [2]sunithamallakula@yahoo.com

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Abstract

In this article, a class of paracontact metric manifolds known as the SP-Kenmotsu (Special Para-Kenmotsu) is considered that accepts a connection of quarter-symmetric metric. In this work it was found out that, SP-Kenmotsu manifold admitting quarter-symmetric metric connection is ξ-concircularly flat if and only when the scalar curvature $\tilde{r}$ with regard to quarter-symmetric metric connection is equal to $2n(n-1)$. We also proved that if the resultant connection of quarter-symmetric metric is ϕ-concircularly flat, it is η-Einstein manifold in terms of quarter-symmetric metric connection. Lastly, in the study example of 5-D SP-Kenmotsu is also included, which confirms the findings presented in this paper.

Keywords: Scalar curvature, Con-circular curvature tensor, Ricci tensor, Quarter-symmetric metric connection, *SP*-Kenmotsu manifold, Einstein manifold.

2010 AMS classification: 53C07, 53C25.

1. Introduction

Sato proposed a concept of almost paracontact Riemannian manifold in [12]. Special para-Sasakian and para-Sasakian manifolds, that is considered as some specific type of almost contact Riemannian manifolds, were defined by Matsumoto and Adati in 1977 [1]. Kenmotsu developed a class of almost contact Riemannian manifolds before Sato [8]. Para or *P*-Kenmotsu and special para or *SP*-Kenmotsu manifolds were described by Sai Prasad and Sinha in 1995 as the class of almost paracontact metric manifolds [13]. Furthermore, numerous geometers have examined these manifolds, including Balga [2], Olszak [9], Srivastava and Srivastava [14], and have derived various findings on them.

On the other hand, the semi-symmetric metric connection plays an important role in the geometry of Riemannian manifolds and also having physical applications. In 1924, Friedmann and Schouten [4] introduced semi-symmetric linear connection on a differentiable manifold and in 1932, Hayden [6] defined it on a Riemannian manifold. As a generalization of semi-symmetric connection, quarter-symmetric connection was introduced by Golab [5]. Later, many geometers have explored quarter-symmetric metric connections on a class of Riemannian manifolds, including Biswas and De [3], Kalpana and Priti Srivastava [7], Prakasha and Taleshian [10], Prakasha [11], Sunitha, Satyanarayana and Sai Prasad [15]. We investigate the geometrical significance of *SP*-Kenmotsu manifolds admitting quarter-symmetric metric connection in the current study, which is motivated by such studies.

The following is the layout of the current paper: Following the introduction, Section 2 includes some preliminaries on para-Kenmotsu manifolds. It is demonstrated in section 3 that *SP*-Kenmotsu manifold admitting quarter-symmetric metric connection is ξ-concircularly flat in case if the scalar curvature equals to $2n(n-1)$ *w. r. t.* quarter-symmetric connection. In addition, we established in section 4 that an *SP*-Kenmotsu manifold admitting a quarter-symmetric metric connection is an η-Einstein manifold with respect to the quarter-symmetric metric connection if it is ϕ-concircularly flat. Finally, in section 5, we provide an example of a 5-D *SP*-Kenmotsu manifold which permits a quarter symmetric metric connection, demonstrating the validity of the results described in this study.

2. Preliminaries

Consider M_n to be an n-dimensional differentiable manifold with structure tensors (ϕ, ξ, η), where the tensor field of type (1, 1) is represented by ϕ, the vector field by ξ, and the 1-form by η such that

$$(a)\quad \eta(\xi) = 1,$$
$$(b)\quad \Phi^2(X) = X - \eta(X)\xi;\ \overline{X} = \Phi X. \tag{2.1}$$

Here, the almost paracontact manifold is represented by M_n manifold.

For all X, Y vector fields on M_n, assume $g(X, Y)$ be Riemannian metric as given by:

$$(a)\quad g(X, \xi) = \eta(X),$$
$$(b)\quad \Phi\xi = 0, \eta(\Phi X) = 0, \mathrm{rank}\,\Phi = n - 1, \tag{2.2}$$
$$(c)\quad g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

Manifold [12] is therefore referred as admitting an almost paracontact Riemannian structure (ϕ, ξ, η, g).

Furthermore, this M_n manifold is known as a para-Kenmotsu (also *P*-Kenmotsu) manifold, if the structure satisfies the conditions given below [13]:

$$(a)(\nabla_X\eta)Y - (\nabla_Y\eta)X = 0,$$
$$(b)(\nabla_X\nabla_Y\eta)Z = [-g(X, Z) + \eta(X)\eta(Z)]\eta(Y) + [-g(X, Y) + \eta(X)\eta(Y)]\eta(Z), \tag{2.3}$$
$$(c)\ \nabla_X\xi = \Phi^2 X = X - \eta(X)\xi,$$
$$(d)(\nabla_X\Phi)Y = -g(X, \Phi Y)\xi - \eta(Y)\Phi X;$$

for every $X, Y, Z \epsilon \chi(M_n)$, and $\chi(M_n)$ is a set of differentiable vector fields M_n. ∇ is the covariant differentiation operator in terms of g, the Riemannian metric.

M_n, a para- Kenmotsu manifold admitting a 1-form η satisfies the equation

$$(a)\quad (\nabla_X\eta)Y = g(X, Y) - \eta(X)\eta(Y) \text{ and}$$
$$(b)\quad (\nabla_X\eta)Y = \varphi(\overline{X}, Y); \tag{2.4}$$

where φ is the associate of ϕ, is referred to as *SP*-Kenmotsu manifold [13].

Assume (M_n, g) to be n-dimensional, $n \geq 3$, manifold differentiable for class C^∞ where the Levi-Civita connection is given by ∇. For tensor *R*, of type (1, 3), the Riemannian Christoffel curvature is then calculated as follows:

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{2.5}$$

S denotes Ricci operator with $S^2(0,2)$ tensor is represented as:

$$g(SX,Y) = S(X,Y), and \quad 2.6$$

$$S^2(X,Y) = S(SX,Y). \quad 2.7$$

For all $X,Y,Z \in \chi(M_n)$, the following relationships are hold in a para-Kenmotsu manifold [13]:

$$\begin{aligned}
&(a) \quad g[R(X,Y)Z,\xi] = \eta[R(X,Y,Z)] = g(X,Z)\eta(Y) - g(Y,Z)\eta(X);\\
&(b) \quad R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X;\\
&(c) \quad R(X,Y)\xi = \eta(Y)X - \eta(X)Y; \text{ when } X \text{ is orthogonal to } \xi;\\
&(d) \quad S(X,\xi) = (n-1)\eta(X);\\
&(e) \quad S(\Phi X,\Phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y).
\end{aligned} \quad 2.8$$

The M_n is deduced to η-Einstein when Ricci tensor attains the form:

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \quad 2.9$$

where a and b are some scalar functions on M_n. Particularly, M_n reduces to an Einstein manifold when $b = 0$.

If in a Riemannian manifold M_n, the $T(X,Y)$ torsion tensor of $\tilde{\nabla}$ linear connection is satisfied [5], then this defines a quarter-symmetric metric connection:

$$T(X,Y) = \eta(Y)\Phi X - \eta(X)\Phi Y, and \quad 2.10$$

$$(\tilde{\nabla}_X g)(Y,Z) = 0. \quad 2.11$$

The torsion tensor (2.10) is provided by a quarter-symmetric metric connection $\tilde{\nabla}$,

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)\,\Phi X \;\; - \varphi(X,Y)\xi \quad 2.12$$

where the Riemannian connection is denoted by ∇.

The quarter-symmetric metric connection $\tilde{\nabla}$, in *SP*-Kenmotsu manifold, is expressed by

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)\bar{X} - 'F(X,Y)\xi. \quad 2.13$$

Let $\tilde{R}$ be the curvature tensor for connections $\tilde{\nabla}$ whereas R be of ∇, correspondingly. Then the relation between these curvature tensors is given as [15]:

$$\tilde{R}(X,Y)Z = R(X,Y)Z + g(Y,Z)X - g(X,Z)Y; \quad 2.14$$

for X, Y and Z vector fields on M_n.

Also, we have [15]

$$\tilde{S}(Y,Z) = S(Y,Z) + ng(Y,Z) - \eta(Y)\eta(Z) \text{ and} \tag{2.15}$$

$$\tilde{r} = r + n^2 - 1, \tag{2.16}$$

for X, Y and Z vector fields on M_n. Here the Ricci tensors are denoted by $\tilde{S}$ and S, and the scalar curvatures be $\tilde{r}$ and r, for $\tilde{\nabla}$ and ∇ connections.

For $Z = \xi$, eq (2.15) becomes:

$$\tilde{S}(Y,\xi) = 2(n-1)\eta(Y). \tag{2.17}$$

Using (2.8) and (2.14), we get

$$\tilde{R}(\xi,X)U = 2[g(X,U)\xi - \eta(U)X], \text{ and} \tag{2.18}$$

$$\tilde{R}(X,Y)\xi = 2[\eta(Y)X - \eta(X)Y]. \tag{2.19}$$

3. ξ-Concircularly flat *SP*-Kenmotsu manifolds admitting quarter-symmetric metric connection

On *n*-dimensional M_n, the con-circular curvature tensor is defined as [16], [17]:

$$\mathrm{W}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{3.1}$$

for all $X, Y, Z \in \chi(M_n)$.

In an *SP*-Kenmotsu manifold, let $\tilde{W}$ be the con-circular curvature tensor with respect to $\tilde{\nabla}$.

Using (2.14) and (3.1), we get

$$\tilde{W}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{\tilde{r}}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{3.2}$$

which specifies the relation in con-circular curvature tensors of *SP*-Kenmotsu manifold *w. r. t.* ∇ and $\tilde{\nabla}$, for X, Y and Z vector fields on M_n.

In (3.2), put $X = \xi$ using (2.18), we get

$$\tilde{W}(\xi,Y)Z = \left[2 - \frac{\tilde{r}}{n(n-1)}\right][g(Y,Z)\xi - \eta(Z)Y]. \tag{3.3}$$

Definition 3.1: If $\widetilde{W}(X,Y)\xi = 0$, an *SP*-Kenmotsu manifold called ξ-concircularly flat [16] with respect to $\tilde{\nabla}$, with $X, Y \in \chi(M_n)$.

Theorem 3.1: If and only if the scalar curvature $\tilde{r}$ with respect to quarter-symmetric metric connection is equal to $2n(n-1)$, an *SP*-Kenmotsu manifold admitting quarter-symmetric metric connection is ξ-concircularly flat.

Proof: From (2.14), equation (3.2) can be expressed as

$$\widetilde{W}(X,Y)Z = R(X,Y)Z + g(Y,Z)X - g(X,Z)Y - \frac{\tilde{r}}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \quad 3.4$$

Substituting $Z = \xi$ and from (2.8), equation (3.4) becomes:

$$\widetilde{W}(X,Y)\xi = \left[2 - \frac{\tilde{r}}{n(n-1)}\right][\eta(Y)X - \eta(X)Y]. \quad 3.5$$

When $\widetilde{W}(X,Y)\xi = 0$, it defines $\tilde{r} = 2n(n-1)$ (or) $\tilde{R}(X,Y)\xi = \eta(Y)X - \eta(X)Y = 0$, which means $\eta(X) = 0$ is not acceptable.

In contrast, if $\tilde{r} = 2n(n-1)$, it follows the equation (3.5) that $\widetilde{W}(X,Y)\xi = 0$ and hence it verifies the theorem proof.

4. Φ-Concircularly flat *SP*-Kenmotsu manifolds admitting quarter-symmetric metric connection

Definition 4.1: If $\widetilde{W}(\Phi X, \Phi Y, \Phi Z, \Phi U) = 0$, an *SP*-Kenmotsu manifold is considered Φ-concircularly flat [16] with respect to $\tilde{\nabla}$ where $X, Y, Z, U \in \chi(M_n)$.

Theorem 4.1: The manifold with respect to quarter-symmetric metric connection is a η-Einstein manifold if an *SP*-Kenmotsu manifold admitting quarter-symmetric metric connection is Φ-concircularly flat.

Proof: Using (3.4), we obtain:

$$\widetilde{W}(X,Y,Z,U) = R(X,Y,Z,U) + g(Y,Z)g(X,U) - g(X,Z)g(Y,U) - \frac{\tilde{r}}{n(n-1)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]. \quad 4.1$$

Substituting $X = \Phi X$, $Y = \Phi Y$, $Z = \Phi Z$, $U = \Phi U$ values in (4.1), we get

$$\begin{aligned}\widetilde{W}(\Phi X,\Phi Y,\Phi Z,\Phi U) &= R(\Phi X,\Phi Y,\Phi Z,\Phi U)\\ &+g(\Phi Y,\Phi Z)g(\Phi X,\Phi U)-g(\Phi X,\Phi Z)g(\Phi Y,\Phi U)\\ &-\frac{\tilde{r}}{n(n-1)}\begin{bmatrix} g(\Phi Y,\Phi Z)g(\Phi X,\Phi U)\\ -g(\Phi X,\Phi Z)g(\Phi Y,\Phi U)\end{bmatrix}.\end{aligned} \tag{4.2}$$

Substituting $X = U = e_i$ in (4.2) and by using summation of period: $i = 1$ to $n - 1$, equation deduced as:

$$\begin{aligned}\widetilde{W}(\Phi e_i,\Phi Y,\Phi Z,\Phi e_i) &= R(\Phi e_i,\Phi Y,\Phi Z,\Phi e_i)\\ &+g(\Phi Y,\Phi Z)g(\Phi e_i,\Phi e_i)-g(\Phi e_i,\Phi Z)g(\Phi Y,\Phi e_i)\\ &-\frac{\tilde{r}}{n(n-1)}\begin{bmatrix} g(\Phi Y,\Phi Z)g(\Phi e_i,\Phi e_i)\\ -g(\Phi e_i,\Phi Z)g(\Phi Y,\Phi e_i)\end{bmatrix};\end{aligned} \tag{4.3}$$

which is equivalent to:

$$\begin{aligned}&\widetilde{W}(\Phi e_i,\Phi Y,\Phi Z,\Phi e_i)\\ &\quad= S(\Phi Y,\Phi Z)-g(\Phi Y,\Phi Z)-\frac{\tilde{r}}{n(n-1)}[0-(n-2)g(\Phi Y,\Phi Z)]\\ &\quad= S(\Phi Y,\Phi Z)-\left[1-\frac{\tilde{r}(n-2)}{n(n-1)}\right]g(\Phi Y,\Phi Z).\end{aligned} \tag{4.4}$$

In view of (2.2) and (2.8), the equation (4.4) becomes

$$\begin{aligned}\widetilde{W}(\Phi e_i,\Phi Y,\Phi Z,\Phi e_i) &= S(Y,Z)+(n-1)\eta(Y)\eta(Z)\\ &-\left[1-\frac{\tilde{r}(n-2)}{n(n-1)}\right][g(Y,Z)-\eta(Y)\eta(Z)].\end{aligned} \tag{4.5}$$

Further on using (2.15), the above equation (4.5) reduces to

$$\begin{aligned}\widetilde{W}(\Phi e_i,\Phi Y,\Phi Z,\Phi e_i) &= \tilde{S}(Y,Z)-\left[(n+1)-\frac{\tilde{r}(n-2)}{n(n-1)}\right]g(Y,Z)\\ &+\left[(n+1)-\frac{\tilde{r}(n-2)}{n(n-1)}\right]\eta(Y)\eta(Z).\end{aligned} \tag{4.6}$$

Now, let us assume that manifold under consideration is Φ-concircularly flat. Then, we get $\widetilde{W}(\Phi e_i,\Phi Y,\Phi Z,\Phi e_i) = 0$.

In view of this, the equation (4.6) deduced as:

$$\tilde{S}(Y,Z) = \left[(n+1)-\frac{\tilde{r}(n-2)}{n(n-1)}\right]g(Y,Z) - \left[(n+1)-\frac{\tilde{r}(n-2)}{n(n-1)}\right]\eta(Y)\eta(Z). \tag{4.7}$$

It is clear that equation (4.7) is of the form:

$\tilde{S}(Y,Z) = \mathrm{a}\, \mathrm{g}(\mathrm{Y},\mathrm{Z}) + \mathrm{b}\, \eta(Y)\eta(Z);$

where,

$a = \left[(n+1) - \frac{\tilde{r}(n-2)}{n(n-1)}\right]$ and $b = -\left[(n+1) - \frac{\tilde{r}(n-2)}{n(n-1)}\right]\eta(Y)\eta(Z)$.

This proves that the manifold under consideration is an η-Einstein manifold.

5. Example of a 5-dimensional *SP*-Kenmotsu manifold admitting quarter-symmetric metric connection

Example 5.1: $M_5 = \{(x, y, z, u, v) \in R^5\}$, is a 5-dimensional manifold, in which (x, y, z, u, v) are the standard coordinates in R^5. Let e_1, e_2, e_3, e_4 and e_5 vector fields on M_5 be as follows:

$$e_1 = e^{-u}\frac{\partial}{\partial x},\quad e_2 = e^{-u}\frac{\partial}{\partial y},\quad e_3 = e^{-u}\frac{\partial}{\partial z},\quad e_4 = e^{-u}\frac{\partial}{\partial v},\quad e_5 = \frac{\partial}{\partial u} = \xi. \qquad 5.1$$

For each point of M_5, $\{e_1, e_2, e_3, e_4, e_5\}$ defines linearly independent vector sets and therefore, forms the basis of $\chi(M_5)$.

Following equation defines the Riemannian metric $g(X, Y)$:

$$g(e_i, e_j) = \begin{cases} 1, & if\ i = j \\ 0, & if\ i \neq j;\ i, j = 1,2,3,4,5. \end{cases}$$

Let η be the 1-form is given as follows:

$\eta(Z) = g(Z, e_5)$, for any $Z \in \chi(M_5)$.

Let Φ be a (1, 1)-tensor field on M_5 with the following definition:

$\Phi^2(e_1) = e_1, \Phi^2(e_2) = e_2, \Phi^2(e_3) = e_3, \Phi^2(e_4) = e_4, \Phi^2(e_5) = 0$, and

$\Phi(e_1) = e_1, \Phi(e_2) = e_2, \Phi(e_3) = e_3, \Phi(e_4) = e_4, \Phi(e_5) = 0$.

The linearity of Φ and $g(X, Y)$ is deduced into:

$\eta(e_5) = 1, \Phi^2(Z) = Z - \eta(Z)e_5$, and $g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y)$;

for $X, Y, Z, U \in \chi(M_5)$.

Hence, for $e_5 = \xi$, an almost paracontact structure on M_5 defined by the structure (ϕ, ξ, η, g) defines.

Applying (5.1), we get

$[e_1, e_5] = e_1, [e_2, e_5] = e_2, [e_3, e_5] = e_3, [e_4, e_5] = e_4;$

$[e_1, e_2] = [e_1, e_3] = [e_1, e_4] = [e_2, e_3] = [e_2, e_4] = [e_3, e_4] = 0.$

For the Levi-Civita connection ∇, the Koszul's formula for metric tensor g is as follows:

$$\begin{aligned}2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y)\\ &-g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]).\end{aligned} \quad 5.2$$

We deduce the following equations by applying the Koszul's formula and considering $e_5 = \xi$;

$$\begin{aligned}&\nabla_{e_1}e_1 = -e_5, \nabla_{e_1}e_2 = 0, \nabla_{e_1}e_3 = 0, \nabla_{e_1}e_4 = 0, \nabla_{e_1}e_5 = e_1;\\ &\nabla_{e_2}e_1 = 0, \nabla_{e_2}e_2 = -e_5, \nabla_{e_2}e_3 = 0, \nabla_{e_2}e_4 = 0, \nabla_{e_2}e_5 = e_2;\\ &\nabla_{e_3}e_1 = 0, \nabla_{e_3}e_2 = 0, \nabla_{e_3}e_3 = -e_5, \nabla_{e_3}e_4 = 0, \nabla_{e_3}e_5 = e_3;\\ &\nabla_{e_4}e_1 = 0, \nabla_{e_4}e_2 = 0, \nabla_{e_4}e_3 = 0, \nabla_{e_4}e_4 = -e_5, \nabla_{e_4}e_5 = e_4;\\ &\nabla_{e_5}e_1 = 0, \nabla_{e_5}e_2 = 0, \nabla_{e_5}e_3 = 0, \nabla_{e_5}e_4 = 0, \nabla_{e_5}e_5 = 0.\end{aligned} \quad 5.3$$

It is clear from the above-mentioned calculations that the manifold under consideration satisfies the properties $\nabla_X\xi = \Phi^2 X = X - \eta(X)\xi$ and $(\nabla_X\Phi)Y = = -g(X,\Phi Y)\xi - \eta(Y)\Phi X$ for all $e_5 = \xi$.

Thus, the manifold M_5 is an *SP*-Kenmotsu manifold with structure tensors (Φ, ξ, η, g).

We now get the following set of values by combining (2.12) and (5.3):

$$\begin{aligned}&\tilde\nabla_{e_1}e_1 = -2e_5, \tilde\nabla_{e_1}e_2 = 0, \tilde\nabla_{e_1}e_3 = 0, \tilde\nabla_{e_1}e_4 = 0, \tilde\nabla_{e_1}e_5 = 2e_1;\\ &\tilde\nabla_{e_2}e_1 = 0, \tilde\nabla_{e_2}e_2 = -2e_5, \tilde\nabla_{e_2}e_3 = 0, \tilde\nabla_{e_2}e_4 = 0, \tilde\nabla_{e_2}e_5 = 2e_2;\\ &\tilde\nabla_{e_3}e_1 = 0, \tilde\nabla_{e_3}e_2 = 0, \tilde\nabla_{e_3}e_3 = -2e_5, \tilde\nabla_{e_3}e_4 = 0, \tilde\nabla_{e_3}e_5 = 2e_3;\\ &\tilde\nabla_{e_4}e_1 = 0, \tilde\nabla_{e_4}e_2 = 0, \tilde\nabla_{e_4}e_3 = 0, \tilde\nabla_{e_4}e_4 = -2e_5, \tilde\nabla_{e_4}e_5 = 2e_4;\\ &\tilde\nabla_{e_5}e_1 = 0, \tilde\nabla_{e_5}e_2 = 0, \tilde\nabla_{e_5}e_3 = 0, \tilde\nabla_{e_5}e_4 = 0, \tilde\nabla_{e_5}e_5 = 0.\end{aligned} \quad 5.4$$

Further, in view of (2.5) and (5.3), the non-vanishing elements of a curvature tensor can be written as:

$$\begin{aligned}&R(e_1,e_2)e_1 = -e_2, R(e_1,e_2)e_2 = e_1, R(e_1,e_3)e_1 = -e_3, R(e_1,e_3)e_3 = e_1;\\ &R(e_1,e_4)e_1 = -e_4, R(e_1,e_4)e_4 = e_1, R(e_1,e_5)e_1 = -e_5, R(e_1,e_5)e_5 = e_1;\\ &R(e_2,e_3)e_2 = -e_3, R(e_2,e_3)e_3 = e_2, R(e_2,e_4)e_2 = -e_4, R(e_2,e_4)e_4 = e_2;\\ &R(e_2,e_5)e_2 = -e_5, R(e_2,e_5)e_5 = e_2, R(e_3,e_4)e_3 = -e_4, R(e_3,e_4)e_4 = e_3;\\ &R(e_3,e_5)e_3 = -e_5, R(e_3,e_5)e_5 = e_3, R(e_4,e_5)e_4 = -e_5, R(e_4,e_5)e_5 = e_4.\end{aligned} \quad 5.5$$

Then by using (5.5), scalar curvatures and Ricci tensors are obtained as:

$S(e_1,e_1) = S(e_2,e_2) = S(e_3,e_3) = S(e_4,e_4) = S(e_5,e_5) = 4$ and $r = 20$,

where $S(X,Y) = \sum_{i=1}^{5} g(R(e_i,X)Y, e_i)$ *and* $r = \sum_{i=1}^{5} S(e_i,e_i)$.

Similarly, by virtue of (2.5), (2.14) and (5.4), we have

$$\begin{aligned}
&\tilde{R}(e_1,e_2)e_1 = -2e_2, \tilde{R}(e_1,e_2)e_2 = 2e_1, \tilde{R}(e_1,e_3)e_1 = -2e_3;\\
&\tilde{R}(e_1,e_3)e_3 = 2e_1, \tilde{R}(e_1,e_4)e_1 = -2e_4, \tilde{R}(e_1,e_4)e_4 = 2e_1;\\
&\tilde{R}(e_1,e_5)e_1 = -2e_5, \tilde{R}(e_1,e_5)e_5 = 2e_1, \tilde{R}(e_2,e_3)e_2 = -2e_3;\\
&\tilde{R}(e_2,e_3)e_3 = 2e_2, \tilde{R}(e_2,e_4)e_2 = -2e_4, \tilde{R}(e_2,e_4)e_4 = 2e_2;\\
&\tilde{R}(e_2,e_5)e_2 = -2e_5, \tilde{R}(e_2,e_5)e_5 = 2e_2, \tilde{R}(e_3,e_4)e_3 = -2e_4;\\
&\tilde{R}(e_3,e_4)e_4 = 2e_3, \tilde{R}(e_3,e_5)e_3 = -2e_5, \tilde{R}(e_3,e_5)e_5 = 2e_3;\\
&\tilde{R}(e_4,e_5)e_4 = -2e_5, \tilde{R}(e_4,e_5)e_5 = 2e_4.
\end{aligned} \qquad 5.6$$

Then by using (5.6), we get the Ricci tensors and scalar curvatures *w. r. t.* quarter-symmetric metric connection expressed as:

$\tilde{S}(e_1,e_1) = \tilde{S}(e_2,e_2) = \tilde{S}(e_3,e_3) = \tilde{S}(e_4,e_4) = \tilde{S}(e_5,e_5) = 8$ and $\tilde{r} = 40$,

where $\tilde{S}(X,Y) = \sum_{i=1}^{5} g\big(\tilde{R}(e_i,X)Y,e_i\big)$ and $\tilde{r} = \sum_{i=1}^{5} \tilde{S}(e_i,e_i)$.

Let X, Y, Z and W be given as:

$$\begin{aligned}
X &= a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5,\\
Y &= b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4 + b_5e_5,\\
Z &= c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4 + c_5e_5,\\
W &= d_1e_1 + d_2e_2 + d_3e_3 + d_4e_4 + d_5e_5;
\end{aligned} \qquad 5.7$$

where, all non-zero real numbers are given by a_i, b_i, c_i, d_i with $i = 1, 2, 3, 4, 5$.

In contrast, substituting $Z = \xi = e_5$ in equation (3.2), it becomes

$$\widetilde{W}(X,Y)\xi = \tilde{R}(X,Y)\xi - \frac{\tilde{r}}{n(n-1)}[\eta(Y)X - \eta(X)Y]. \qquad 5.8$$

Now, by substituting the defined values of tensor fields X and Y from (5.7) in the equation (5.8) and by putting $\tilde{r} = 40$, it becomes

$$\begin{aligned}
\widetilde{W}(X,Y)\xi \quad &= (2e_1 - 2e_1)a_1b_5 + (2e_2 - 2e_2)a_2b_5 + (2e_3 - 2e_3)a_3b_5\\
&+(2e_4 - 2e_4)a_4b_5 + (-2e_1 + 2e_1)a_5b_1 + (-2e_2 + 2e_2)a_5b_2\\
&+(-2e_3 + 2e_3)a_5b_3 + (-2e_4 + 2e_4)a_5b_4 = 0.
\end{aligned} \qquad 5.9$$

This confirms the finding mentioned in Section 3.

We now show that M_5 manifold, which is under consideration and shown previously as *SP*-Kenmotsu manifold, is Φ-concircularly flat *w. r. t.* $\tilde{\nabla}$. That is, we have to demonstrate $\widetilde{W}(\Phi X, \Phi Y, \Phi Z, \Phi U) = 0$.

Using (3.4), we obtain

$$\begin{aligned}\widetilde{W}(\Phi X,\Phi Y,\Phi Z,\Phi U) &= R(\Phi X,\Phi Y,\Phi Z,\Phi U) \\ &+g(\Phi Y,\Phi Z)g(\Phi X,\Phi U)-g(\Phi X,\Phi Z)g(\Phi Y,\Phi U) \\ &-\frac{\tilde{r}}{n(n-1)}\begin{bmatrix} g(\Phi Y,\Phi Z)g(\Phi X,\Phi U) \\ -g(\Phi X,\Phi Z)g(\Phi Y,\Phi U)\end{bmatrix}.\end{aligned} \quad 5.10$$

Then, by virtue of (5.7), equation (5.10) reduces to:

$$\begin{aligned}&\widetilde{W}(\Phi X,\Phi Y,\Phi Z,\Phi U) \\ &= 2(a_1b_2-a_2b_1)(c_1d_2-c_2d_1) + 2(a_1b_3-a_3b_1)(c_3d_1-c_1d_3) \\ &+ 2(a_1b_4-a_4b_1)(c_4d_1-c_1d_4) + 2(a_2b_3-a_3b_2)(c_3d_2-c_2d_3) \\ &+ 2(a_2b_4-a_4b_2)(c_2d_4-c_4d_2) + 2(a_3b_4-a_4b_3)(c_4d_3-c_3d_4) \\ &= 0.\end{aligned} \quad 5.11$$

We now have proof that manifold M_5 of the *SP*-Kenmotsu manifold is Φ-concircularly flat for quarter-symmetric metric connections: $\frac{c_1}{d_1}=\frac{c_2}{d_2}=\frac{c_3}{d_3}=\frac{c_4}{d_4}$.

The outcomes mentioned in Section 4 are supported by the evidence presented in this section.

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Conflict of interests

None

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