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# On Approximation of a function f ϵ W (L $_p$, $\xi$(t)) Class by (C, 1) [F, $d_n$] Means of its Fourier Series

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**Abstract**

The [F, $d_n$] matrix method were introduced by Jakimovsky as generalization both the Euler $E_r$ method and Stirling-Karamata-Lototsky method. Where $d_n$ be a fixed sequence of positive number and the class $W(L_p, (\xi(t))$ equal to $Lip(\xi(t), p)$ for β = 0 and since [F, $d_n$] method includes (E, q) method then using Cesąro mean and [F, $d_n$] mean associate with infinite series. In this paper, firstly we have improved a theorem of Nigam under weaker conditions and then we have defined (C, 1) [F, $d_n$] mean.

***Keywords:*** (C, 1) mean, Fourier series, W ($L_p$, $\xi$(t)) class, Lebesgue integral (C, 1) [F, $d_n$] summability, [F, $d_n$] mean.

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## 1. Introduction

The summability method [F, $d_n$] was pronounced by Jakimovsky [4]. Further extended on approximation of f belong to many classes also W($L_p$, ξ(t)) by Cesąro mean, Nörlund mean has been discussed by investigator like respectively Alexits [1], Chandra [2], Sahney and Geol [18], Khan [6], Quereshi [17], Deepmala et al. [3], Mishra et al. ([11],[12],[13],[14]), Shrivastava, verma and yadav [21] studied the "Approximation of function of a class Lip (α, p) by [F, $d_n$] mean". Shrivastava and Rathore [20] extended this result using on "Approximation of a function of class W $(L_p,(\xi(t))$ by [F, $d_n$] mean of Fourier series". Further has been also studied about product summability on approximation has been obtained by several researchers like Lal and Singh [8] and Lal and Kushwaha [7]. Recently Nigam [16] have determined on approximation of a function belonging to W$\left(L_r,\xi(t)\right)$ by (C, 1) (E, q) product summability. But till now no work done to extend the result on approximation of f ϵ W$\left(L_p,\xi(t)\right)$ with (C, 1) [F, $d_n$] mean has been proved.

## 2. **Definition and Notation**

Let f(x) be a periodic function and Lebesgue integrable on [-π, π].Then the Fourier series of f(x) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n cosnx + b_n sinnx) \tag{2.1}$$

Then

$$\phi(x) = \frac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\}. \tag{2.2}$$

Let $d_1$, $d_2$------$d_n$, be a fixed sequence of positive number and x be a real number. The element $P_{nk}$ of [F, $d_n$] matrix are defined by the relations

$$\prod_{j=1}^{n}\frac{x+d_j}{1+d_j} = \sum_{k=0}^{\infty} P_{nk}\, x^k \tag{2.3}$$

And $$P_{00} = 1 . \tag{2.4}$$

Let $$\sigma_x\,(f; x) = \sum_{k-0}^{\infty} P_{nk}\, S_k\,(f; x). \tag{2.5}$$

Denote the [F. $d_n$] mean of f ϵ L [-π, π] at x, where $S_k$ (f; x) is the $k^{th}$ partial sum of (2.1).The [F, $d_n$] method were introduced by Jakimovsky [4] as generalization of both the Euler $E_r$ method and Stirling- Karamata-Lototsky method. When $d_n = \frac{(n-1)}{c}$, n= 1, 2, 3------, c, a positive integer the [F, $d_n$] matrix reduces to the matrix corresponding to the Stirling-Karamata-Lototsky method defined by Karamata [5]. The Euler $E_r$ (0<r<1) are obtained with $d_n = \frac{(1-r)}{r}$ , n =1, 2, 3---- Lorch and Newman [9] studied the Lebesgue constant for this method. Several fundamental properties of [F, $d_n$] matrix have been discussed in Meir [10] and Miracle [15]

$$S_n = 2\sum_{k=1}^{n} \frac{d_k}{(1+d_k)^2} \tag{2.6}$$

and

$$U_n = 1 + 2\sum_{k=1}^{n} \frac{1}{(1+d_k)} \tag{2.7}$$

The [F, $d_n$] matrix is regular by Jakimovsky [4] if $U_n \to \infty$ as $n \to \infty$ we shall consider only regular matrices and indeed assume that for large n then $d_n$ is bounded away from zero.

The $\frac{d_n}{(1+d_n)^2}$ is also bounded and $S_n \to \infty$ as $n \to \infty$

n + 1 = [$U_n$] be the integral part of $U_n$

The (C, 1) method is defined the series $\sum_{k=0}^{\infty} u_k$ is summable to S with $n^{th}$ partial sum is given by

If $$(C, 1) = \frac{1}{(n+1)} \sum_{k=1}^{n} S_k \quad \to S \text{ as } n \to \infty \tag{2.8}$$

The (C, 1) and [F, $d_n$] mean is defines product (C, 1) [F, $d_n$] of the partial sum $S_n$ of $\sum_{k=0}^{\infty} u_k$ .

Then $$(C^1, F)_n^{d_n} = \frac{1}{(n+1)} \sum_{k=0}^{n} F_k^{d_n} \to S \text{ as } n \to \infty \tag{2.9}$$

where $F_n^{d_n}$ denotes the [F, $d_n$] transform of $S_n$ then $\sum_{k=0}^{\infty} u_k$ is summable (C, 1) [F, $d_n$] transform to S.

If f ∈ Lip α

$$|f(x+t) - f(x)| = O\left(|t|^{\alpha}\right), \quad \text{for } 0 < \alpha \le 1$$

Similarly Lip α subset of Lip(α, p) subset of Lip(ξ(t), p) and the function $\xi(t)$ is positive increasing and f ∈W$(L_P,(\xi(t))$ then

$$\left(\int_0^{2\pi}\left|\{f(x+t)-f(x)\}\sin^{\beta}x\right|^{P}dx\right)^{1/P}=\mathrm{O}\left(\xi(t)\right),\qquad \beta\geq 0$$

The W$(L_P,(\xi(t))$ equal to $Lip(\xi(t),p)$ for β = 0. Let the degree of approximation $E_n(f)$ be given by

$$E_n(f)=\min\|f-T_n\|_p,\ \text{(Zygmund [22])} \tag{2.10}$$

where T $_n$(x) is a trigonometric polynomial of degree n.

**3. Some Theorems:** G. Alexits [1] proved the following theorem.

**Theorem A-** If f ∈ Lip α, 0<α≤ 1 be a periodic function then the degree of approximation of the $(C,\delta)$ means of its Fourier series for $0<\alpha<\delta\leq 1$is given by

$$\max_{0\leq x\leq 2\pi}\left|f(x)-\sigma_n^{\delta}(x)\right|=O\left(\frac{1}{n^{\alpha}}\right), \tag{3.1}$$

and for $0<\alpha\leq\delta\leq 1$ is given by

$$\max_{0\leq x\leq 2\pi}\left|f(x)-\sigma_n^{\delta}(x)\right|=O\left(\frac{\log\ n}{n^{\alpha}}\right), \tag{3.2}$$

where $\sigma_n^{\delta}$ are the partial sums of (2.1) its mean (C, δ).

H. K. Nigam [16] obtained, on degree of approximation of a function $f\in W\left(L_r,\xi(t)\right)$ class by (C, 1) (E, q) product summability mean of Fourier series is given by

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**Theorem B**: If $f$ is a 2π- periodic function and Lebesgue integrable on [0, 2π] and belongs to $W(L_r, \xi(t))$ class, then its degree of approximation is given by

$$\left\| C_n^1 E_n^q - f \right\|_r = O\left[ (n+1)^{\beta + 1/r} \xi\left( \frac{1}{n+1} \right) \right] \tag{3.3}$$

provided ξ(t) satisfies the following conditions :

$$\left\{ \frac{\xi(t)}{t} \right\} \text{ be a decreasing sequence.} \tag{3.4}$$

$$\left[ \int_0^{\frac{1}{n+1}} \left\{ \frac{t\,|\,\phi(t)\,|\,sin^{\beta} t}{\xi(t)} \right\}^r dt \right]^{1/r} = O\left( \frac{1}{n+1} \right) \tag{3.5}$$

and $$\left[ \int_{\frac{1}{n+1}}^{\pi} \left\{ \frac{\xi(t)}{t^{-\delta} sin^{\beta} t} \right\}^r dt \right]^{1/r} = O\left\{ (n+1)^{\delta} \right\}, \tag{3.6}$$

where $\delta$ is an arbitrary number such that $s(1-\delta)-1>0$, $\frac{1}{r}+\frac{1}{s}=1$ condition (3.5) and (3.6) hold uniformly in $x$ and $C_n^1 E_n^q$ is (C, 1)(E, q) means of the Fourier series (2.1).

## 4. Main theorem

In this direction, further extended our theorem on approximation of a f ϵ $W(L_p, \xi(t))$ by (C, 1)[F, $d_n$] mean of its Fourier series, has been proved.

**Theorem:** If $f : R \to R$ is periodic function and integrable on [0, 2π] and belonging to $W(L_p, \xi(t))$ class associate with the approximation of $f$ by (C, 1) [F, $d_n$] summability means of its Fourier series (2.1) satisfies.

$$\left| (C^1, F)_n^{d_n} - f(x) \right|_p = O\left\{ (n+1)^{\beta + 1/p}\ \xi\left( \frac{1}{(n+1)} \right) \right\}, \tag{4.1}$$

provided $\xi(t)$ satisfies the following conditions:

$$\left\{\frac{\xi(t)}{t}\right\} \text{ be a decreasing sequence,} \tag{4.2}$$

$$\left[\int_0^{\frac{\pi}{n+1}}\left\{\frac{t|\phi(t)|\sin^{\beta}t}{\xi(t)}\right\}^p dt\right]^{1/p} = O\left(\frac{1}{n+1}\right) \tag{4.3}$$

$$\left[\int_{\frac{\pi}{n+1}}^{\pi}\left\{\frac{\xi(t)}{t^{-\delta}\sin^{\beta}t}\right\}^p dt\right]^{1/p} = O\left\{(n+1)^{\delta}\right\} \tag{4.4}$$

where δ is an arbitrary number such that $q(1-\delta)-1>0$, $\frac{1}{p}+\frac{1}{q}=1$, condition (4.3) and(4.4) hold uniformly in x and($C^1$, $F)_n^{d_n}$ is (C, 1) [F, $d_n$] means of Fourier series (2.1).

5. **Lemmas**: Following lemmas are required for the proof of our theorem:

**Lemma 1**.

$$\prod_{k=1}^{n}\frac{\exp(it)+d_k}{1+d_k} = \exp\{(U_n-1)it/2 - S_n\, t^2/4\} + O(S_n t^3). \tag{5.1}$$

This is due to Lorch and Newman [9]

**Lemma 2.** $K_n(t) = \frac{1}{(n+1)\pi}\sum_{k=0}^{n}\left[\frac{\sin\left(\frac{U_n t}{2}\right)\exp\left(\frac{-S_n\, t^2}{4}\right)}{\sin t_{/2}}\right]$

$= O\,(n+1)$ for $0\le t\le\frac{\pi}{(n+1)}$.

**Proof-** Apply $\sin n t \le n \sin t$ for $0\le t\le\frac{\pi}{n+1}$

Then $K_n(t) = \frac{1}{(n+1)\pi}\sum_{k=0}^{n}\exp\left(\frac{-S_n t^2}{4}\right)\frac{U_n \sin t_{/2}}{\sin t_{/2}}$

$= O\,(U_n)\frac{1}{(n+1)\pi}\sum_{k=0}^{n} 1$

$= O\,(n+1)$. (5.2)

**Lemma 3.** $K_n(t) = \frac{1}{(n+1)\pi}\sum_{k=0}^{n}\left[\frac{\sin\left(\frac{U_n t}{2}\right)\exp\left(\frac{-S_n\, t^2}{4}\right)}{\sin t_{/2}}\right]$. Then $K_n(t) = O\left(\frac{1}{t}\right)$

**Proof**- Using $\sin\frac{t}{2} \geq \left(\frac{t}{\pi}\right)$ and $|sin\frac{U_n t}{2}| \leq 1$ for $\frac{\pi}{n+1} \leq t \leq \pi$

$$K_n(t) = \frac{1}{(n+1)\pi}\sum_{k=0}^{n} \exp\left(\frac{-S_n t^2}{4}\right) \frac{1}{t_{/\pi}}$$

$$= O\left(\frac{1}{t}\right) \tag{5.3}$$

**Lemma** 4. $K_n(t) = \frac{1}{(n+1)\pi}\sum_{k=0}^{n} \frac{O(S_n t^3)}{sin\frac{t}{2}}$

$$= O\,(n+1)$$

**Proof**- Using $\left|\sin\frac{t}{2}\right| \leq \frac{t}{2}$ for $0 \leq t \leq \frac{\pi}{n+1}$

Then $K_n(t) = \frac{1}{(n+1)\pi}\sum_{k=0}^{n} \frac{O(S_n t^3)}{sin\frac{t}{2}}$

$$= \frac{O(S_n)}{(n+1)\pi}\sum_{k=0}^{n} \frac{t^3}{t_{/2}}$$

$$= \frac{O(S_n)}{(n+1)\pi}\frac{n(n+1)(2n+1)}{6}$$

$$= O\,(n+1) \tag{5.4}$$

**Lemma 5**. $K_n(t) = \frac{1}{(n+1)\pi}\sum_{k=0}^{n} \frac{O(S_n t^3)}{sin\frac{t}{2}}$

$$= O\,(n+1)$$

**Proof** - Using $\sin\frac{t}{2} \geq \frac{t}{\pi}$ for $\frac{\pi}{(n+1)} \leq t \leq \pi$

Then $K_n(t) = \frac{1}{(n+1)\pi}\sum_{k=0}^{n} \frac{O(S_n t^3)}{sin\frac{t}{2}}$

$$= \frac{O(S_n)}{(n+1)\pi}\sum_{k=0}^{n} \frac{t^3}{t_{/\pi}}$$

$$= O\,(n+1) \tag{5.5}$$

6. **Proof of the main theorem-**

Titchmarsh [19] and the $k^{th}$ partial sum $S_k(f;x)$ of Fourier series (2.1) is given by

$$S_k(f;x)-f(x) = \frac{1}{\pi}\int_0^{\pi}\frac{1}{\sin t/2}\phi(t)\sin(k+\tfrac{1}{2})t\,dt \tag{6.1}$$

The [F, $d_n$] transform $F_n^{d_n}$ of $S_k(f;x)$ is given by

$$F_n^{d_n} - f(x) = \frac{1}{\pi}\int_0^{\pi}\frac{\phi(t)}{\sin t/2}\sum_{k=0}^{\infty}P_{nk}\sin(k+\tfrac{1}{2})t\,dt \tag{6.2}$$

The (C, 1)[F, $d_n$] transform of $S_k(f;x)$ by $(C^1, F)_n^{d_n}$ then

$$(C^1, F)_n^{d_n} - f(x) = \frac{1}{(n+1)\pi}\sum_{k=0}^{n}\int_0^{\pi}\frac{\phi(x)}{\sin\frac{t}{2}}\sum_{k=0}^{\infty}P_{nk}\sin(k+\tfrac{1}{2})t\,dt \tag{6.3}$$

$$= \frac{1}{(n+1)\pi}\sum_{k=0}^{n}\int_0^{\pi}\frac{\phi(x)}{\sin\frac{t}{2}}\, I_m\left\{\sum_{k=0}^{\infty}P_{nk}\,exp\left(i(k+\tfrac{1}{2})\right)\right\}dt$$

$$= \frac{1}{(n+1)\pi}\sum_{k=0}^{n}\int_0^{\pi}\frac{\phi(x)}{\sin\frac{t}{2}}\, I_m\left\{exp\left(\frac{it}{2}\right)\sum_{k=0}^{\infty}P_{nk}\,exp(ikt)\right\}dt$$

$$= \frac{1}{(n+1)\pi}\sum_{k=0}^{n}\int_0^{\pi}\frac{\phi(x)}{\sin\frac{t}{2}}\, I_m\left\{exp\left(\frac{it}{2}\right)\prod_{k=1}^{n}\frac{exp(it)+d_k}{1+d_k}\right\}dt$$

$$\left|(C^1, F)_n^{d_n} - f(x)\right|$$

$$= \left|\frac{1}{(n+1)\pi}\sum_{k=0}^{n}\int_0^{\pi}\frac{\phi(x)}{\sin\frac{t}{2}}\, I_m\left\{exp\left(\frac{it}{2}\right)\prod_{k=1}^{n}\frac{exp(it)+d_k}{1+d_k}\right\}dt\right. \tag{6.4}$$

$$\leq \frac{1}{(n+1)\pi}\sum_{k=0}^{n}\int_0^{\pi}\frac{|\phi(x)|}{|\sin\frac{t}{2}|}\, I_m\left\{exp\left(\frac{it}{2}\right)\prod_{k=1}^{n}\frac{exp(it)+d_k}{1+d_k}\right\}dt$$

$$= \frac{1}{(n+1)\pi}\sum_{k=0}^{n}\int_0^{\pi}\frac{\phi(x)}{\sin\frac{t}{2}}\left\{I_m\left\{exp\left(\frac{it}{2}\right)\{\exp\left\{\frac{(U_n-1)it}{2}-\frac{S_n\,t^2}{4}\right\}\right\}+O(S_n t^3)\right\}dt$$

$$= \frac{1}{(n+1)\pi}\sum_{k=0}^{n}\int_0^{\pi}\frac{\phi(x)}{\sin\frac{t}{2}}\left\{\sin\left(\frac{U_n t}{2}\right)\exp\left(\frac{-S_n t^2}{4}\right)+O(S_n t^3)\right\}$$

$$\leq \left|\frac{1}{(n+1)\pi}\sum_{k=0}^{n}\int_0^{\pi}\frac{\phi(x)}{\sin\frac{t}{2}}\sin\left(\frac{U_n t}{2}\right)\exp\left(\frac{-S_n t^2}{4}\right)dt\right|$$

$$+\left|\frac{1}{(n+1)\pi}\sum_{k=0}^{n}\int_0^{\pi}\frac{\phi(x)}{sin\frac{t}{2}}\{O(S_n t^3)\}\right|$$

$$= |I_1| + |I_2| \tag{6.5}$$

Then $|I_1| = \frac{1}{(n+1)\pi}\sum_{k=0}^{n}\int_0^{\pi} \frac{|\phi(x)|}{\sin\frac{t}{2}} \sin\left(\frac{U_n t}{2}\right) \exp\left(\frac{-S_n t^2}{4}\right) dt.$

Applying Lemma 2

$$|I_1| = \int_0^{\pi} |\phi(x)| K_n(x) dt$$

$$= \left[\int_0^{\frac{\pi}{n+1}} + \int_{\frac{\pi}{n+1}}^{\pi}.\right] |\phi(x)| K_n(x) dt$$

$$= I_{1.1} + I_{1.2} \tag{6.6}$$

Now $I_{1.1} = \int_0^{\frac{\pi}{n+1}} |\phi(x)| K_n(x) dt$

Applying Hölder's inequality

$$= \left[\int_0^{\frac{\pi}{n+1}} \left\{\frac{t|\phi(t)|\sin^{\beta} t}{\xi(t)}\right\}^p dt\right]^{1/p} \left[\int_0^{\frac{\pi}{n+1}} \left\{\frac{K_n(t)\xi(t)}{t\sin^{\beta} t}\right\}^q dt\right]^{1/q}$$

$$= O\left(\frac{1}{(n+1)}\right). O(n+1) \left[\int_0^{\frac{\pi}{n+1}} \left\{\frac{\xi(t)}{t^{\beta+1}}\right\}^q dt\right]^{1/q}$$

$$= \xi\left(\frac{1}{(n+1)}\right)\left[\int_0^{\frac{\pi}{n+1}} \{t^{-(\beta+1)}\}^q dt\right]^{1/q}$$

$$= \xi\left(\frac{1}{(n+1)}\right)\left[\int_0^{\frac{\pi}{n+1}} t^{-(\beta+1)q} dt\right]^{1/q}$$

$$= O(n+1)^{\beta+1/p}\ \xi\left(\frac{1}{(n+1)}\right) \tag{6.7}$$

Then $I_{1.2} = \int_{\frac{\pi}{n+1}}^{\pi}. |\phi(x)| K_n(x) dt.$

Applying Hölder's inequality

$$= \left[\int_{\frac{\pi}{n+1}}^{\pi} \left\{\frac{t^{-\delta}|\phi(t)|\sin^{\beta} t}{\xi(t)}\right\}^p dt\right]^{1/p} \left[\int_{\frac{\pi}{n+1}}^{\pi} \left\{\frac{K_n(t)\xi(t)}{t^{-\delta}\sin^{\beta} t}\right\}^q dt\right]^{1/q}$$

$$= O(n+1)^{\delta} \left[\int_{\frac{\pi}{n+1}}^{\pi} \left\{\frac{\xi(t)}{t^{-\delta}\, t\sin^{\beta} t}\right\}^q dt\right]^{1/q}$$

$$= O(n+1)^{\delta} \quad \xi\left(\frac{1}{(n+1)}\right) \; \left[ \int_{\frac{\pi}{n+1}}^{\pi} \left\{\frac{\xi(t)}{t^{\beta+1-\delta}}\right\}^{q} dt \right]^{1/q}$$

$$= O(n+1)^{\delta} \quad \xi\left(\frac{1}{(n+1)}\right) \; \left[ \int_{1/\pi}^{n+1/\pi} \left\{\frac{\xi(1/y)}{y^{-\beta-1+\delta}}\right\}^{q} \frac{dy}{y^2} \right]^{1/q} \quad \because t = \frac{1}{y}$$

$$= O(n+1)^{\beta+1/p} \, \xi\left(\frac{1}{(n+1)}\right) \tag{6.8}$$

Now

$$|\, I_2 \,| = \left[\int_0^{\frac{\pi}{n+1}} + \int_{\frac{\pi}{n+1}}^{\pi}.\right] |\phi(x)| K_n(x) dt$$

$$= I_{2.1} + I_{2.2} \tag{6.9}$$

Similarly

$$I_{2.1} = O(n+1)^{\beta+1/p} \; \xi\left(\frac{1}{(n+1)}\right) \tag{6.10}$$

Now

$$I_{2.2} = \int_{\frac{\pi}{n+1}}^{\pi}. |\phi(x)| K_n(x) \, dt$$

Applying Hölder’s inequality

$$= \left[ \int_{\frac{\pi}{n+1}}^{\pi} \left\{\frac{t^{-\delta}|\phi(t)| \sin^{\beta} t}{\xi(t)}\right\}^{p} dt\right]^{1/p} \left[ \int_{\frac{\pi}{n+1}}^{\pi} \left\{\frac{K_n(t)\xi(t)}{t^{-\delta} \sin^{\beta} t}\right\}^{q} dt \right]^{1/q}$$

$$= O\,(n+1)^{\delta} \,.\, O\,(n+1) \left[ \int_{\frac{\pi}{n+1}}^{\pi} \left\{\frac{\xi(t)}{t^{-\delta} \sin^{\beta} t}\right\}^{q} dt \right]^{1/q}$$

$$= O\,(n+1)^{\delta+1} \quad \xi\left(\frac{1}{(n+1)}\right) \; [t^{(\delta-\beta)+1/q}]_{\pi/n+1}^{\pi}$$

$$= O(n+1)^{\beta+1/p} \quad \xi\left(\frac{1}{(n+1)}\right) \tag{6.11}$$

Thus combining (6.5) to (6.11) we set

$$|\,(C^1, F)_n^{d_n} - f(x)\,| = O\left\{(n+1)^{\beta+1/p} \, \xi\left(\frac{1}{(n+1)}\right)\right\}$$

$$\|(C^1, F)_n^{d_n} - f(x)\|_p = \left\{\int_0^{2\pi} O\left\{(n+1)^{\beta+1/p}\,\xi\left(\frac{1}{(n+1)}\right)\right\}^p dx\right\}^{1/p}$$

This completes the proof of the main theorem.

**Application**: Following corollaries can be derived from our main theorem:

**Corollary1.** If β = 0, and $\xi(t) = t^\alpha$, then the degree of approximation of a function $f \in Lip(\alpha, p),\ 0 < \alpha \le 1$ is given by

$$\|(C^1, F)_n^{d_n} - f(x)\|_p = O\left\{\frac{1}{(n+1)^{\alpha-1/p}}\right\}$$

**Corollary2.** If $p \to \infty$ and condition from corollaries 1.

$$\|(C^1, F)_n^{d_n} - f(x)\|_\infty = O\left\{\frac{1}{(n+1)^{\alpha}}\right\}$$

## 7. Conclusion

We would like to mention that the matrix [F, $d_n$] mean and under weaker condition Lip (α, p) given by Shrivastava, Verma and yadav [21] was generalized in 1997. Further our result in generalizing and newer method of summability like F(a, q), (f, $d_n$) (e, c) and Nörlund mean to summable infinite series. Our result in above mentioned special cases as well.

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