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FRACTIONAL DERIVATIVE OPERATORS INVOLVING THE MULTIVARIABLE *H*-FUNCTIONS

Priyanka Gupta[1], Neelam Pandey[2] and Shiv Kant Tiwari[3*]

[1]*Department of Mathematical Sciences, A. P. S. University, Rewa (M.P.)-486003, India*

[2]*Department of Mathematics, Govt. Model Science College, Rewa (M.P.)-486001, India*

[3]*Department of Mathematics, Lukhdhirji Engineering College, Morbi (Gujarat)-363642, India*

Email: [1]priyanka103gupta@gmail.com, [2]dr.pandeyneelam@gmail.com

[3*]shivkant.math@gmail.com (Corresponding author)

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Abstract

The wholesome theme of this present paper is to derive certain interesting results of general class of polynomials and its relation with Saigö fractional operators of multivariable H-functions, as the author has tried his best. Very fast, we established two results that given the product of multivariable H-functions and a general class of polynomials that have been given in Saigö operators. Because of the general nature of the Saigö operator, a general class of polynomials and multivariable H-functions, a large number of new and well-known results involving Riemann-Liouville and Erdélyi-Kober, fractional differential operators and several special functions notably generalized Mittag-Leffler function and Wright hypergeometric function, Whittaker function, etc. follow as special cases of our main result.

Keywords: Saigö Fractional differential operators, Erdélyi-Kober, General Polynomials and Riemann-Liouville, Multivariable *H*-functions, and Mittag-Leffler functions.

2010 AMS classification: 33C70, 26A33, 33C60, 33C45

1. Introduction and Preliminaries

Fractional differential operators involving special functions have found significant importance and applications in various sub-field of applicable mathematical analysis. In the last four decades, some workers like Love [7], McBride [5], Ram and Kumar [13], Khan [6], Kalla and Saxena [21], Saigö [18], Kilbas and Sebastian [4], Saxena et al. [19], Kiryakova [24,25] and Kilbas [1], etc., have studied in the depth of properties, applications and different extensions of various hypergeometric operators of fractional differentiation. A detailed account of such operators along with their properties and applications can be found in the research monographs by Samko, Kilbas and Marichev [22], Miller and Ross [14], Kiryalova [24,25], Kilbas, Srivastava, and Trujillo [9] and Debnath and Bhatta [1]. Generalization of the hypergeometric fractional differentials, including the Saigö operator [16, 17, 18], has been introduced by Samko et al. [22], and Kilbas and Saigö [3] as follows: The multivariable *H*-functions, introduced by Srivastava and Panda [8], is an extension of the multivariable *G*-functions. The multivariable *H*-functions includes Fox's *H*-function, Meijer's *G*-function, the generalized Lauricella function of Srivastava and Daoust [11], Apell function, the Whittaker function as so on. The multivariable *H*-functions is defined and represented in the following manner:

$$H[z_1,\dots\dots,z_r]=H^{0,n:\,m_1,n_1;\,\dots;\,m_r,n_r}_{p,q:\,p_1,q_1;\,\dots;\,p_r,q_r}\left[\begin{matrix} z_1\\ \vdots\\ z_r\end{matrix}\left|\begin{matrix}\left(a_j,\alpha_j^{(1)},\dots,\alpha_j^{(r)}\right)_{1,p}:\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,p_1};\dots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r}\\ \left(b_j,\beta_j^{(1)},\dots,\beta_j^{(r)}\right)_{1,q}:\left(d_j^{(1)},\delta_j^{(1)}\right)_{1,q_1};\dots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r}\end{matrix}\right.\right]$$

$$= \frac{1}{(2\pi i)^r}\int_{L_1}\cdots\int_{L_r}\phi_i(\xi_1)\dots\phi_r(\xi_r)\,\psi(\xi_1,\dots,\xi_r)z_1^{\xi_1},\dots,z_r^{\xi_r}d\xi_1\dots d\xi_r,\quad (1.1)$$

where $i=\sqrt{-1}$ and

$$\psi(\xi_1,\dots,\xi_r)=\frac{\prod_{j=1}^{n}\Gamma\left(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}\xi_i\right)}{\prod_{j=n+1}^{p}\Gamma\left(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}\xi_i\right)\prod_{j=1}^{q}\Gamma\left(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}\xi_i\right)},$$

$$\beta_j^{(1)}=\beta_j^{(\prime)}\quad,\beta_j^{(2)}=\beta_j^{(\prime\prime)}\qquad (1.2)$$

$$\phi_i(\xi_i)\;=\frac{\prod_{j=1}^{m_i}\Gamma\left(d_j^{(i)}-\delta_j^{(i)}\xi_i\right)\prod_{j=1}^{n_i}\Gamma\left(1-c_j^{(i)}+\gamma_j^{(i)}\xi_i\right)}{\prod_{j=m_i+1}^{q_i}\Gamma\left(1-d_j^{(i)}+\delta_j^{(i)}\xi_i\right)\prod_{j=n_i+1}^{p_i}\Gamma\left(c_j^{(i)}-\gamma_j^{(i)}\xi_i\right)},\ \text{for all } i\in(1,\dots,r).\ (1.3)$$

The *M*-Series is a particular case of $\overline{H}$-function of Inayat Hussain. Some properties and applications of *M*-Series and its special cases have been studied by [1], [3], [14].

Comprehensive M-function play an important role in the solution of integral equations and fractional order differential. Recently, generalized of M-Series as the function is defined by in terms of the power series

$$ {}_{p}^{\alpha}M_{q}^{\beta}\left(a_1,\dots,a_p;b_1,\dots,b_q;z\right) = {}_{p}^{\alpha}M_{q}^{\beta}(z) = \sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{z^k}{\Gamma(\alpha k+\beta)}, \quad (1.4) $$

$z,\alpha,\beta \in \mathbb{C}, \Re(\alpha) > 0,$ where, $z,\vartheta,\mu \in \mathbb{C}, \Re(\vartheta) > 0$ *and* $(a_j)_k$ *and* $(b_j)_k$ are known as Pochhammer symbols.

2. Fractional Calculus Operators

Let The generalized fractional differential operators were defined and investigated by Saigö [18] as follow

$$ \left(D_{0+}^{\alpha,\beta,\eta}f\right)(x) = \left(\frac{d}{dx}\right)^n\left(I_{0+}^{(-\alpha+n,-\beta-n,\alpha+\eta-n)}f\right)(x) \quad (2.1) $$

$$ (0 \le Re(\alpha), n = [Re(\alpha)] + 1) $$

$$ \left(D_{-}^{\alpha,\beta,\eta}f\right)(x) = \left(\frac{-d}{dx}\right)^n\left(I_{-}^{(-\alpha+n,-\beta-n,\alpha+\eta)}f\right)(x) \quad (2.2) $$

$$ (0 \le Re(\alpha), n = [Re(\alpha)] + 1) $$

where $\alpha,\beta,\eta \in \mathbb{C}, 0 < Re(\alpha)$ and $\left(D_{0+}^{\alpha,\beta,\eta}\right)$, $\left(D_{-}^{\alpha,\beta,\eta}\right)$ are known to us as fractional differential generalized operators presented by Saigö [18].

When $\beta = -\alpha$ the above operators (1.1) and (1.2) reduce to the following classical Riemann-Liouville fractional differential operators of order $\alpha \in \mathbb{C}, (Re(\alpha) \ge 0)$[2, p. 80, Eqs. (2.2.3), (2.2.4)]:

$$ \left(D_{0+}^{\alpha,-\alpha,\eta}f\right)(x) = \{D_{0+}^{\alpha}f\}\{x\} \equiv \left\{\frac{d}{dx}\right\}^n\frac{1}{\Gamma\{n-\alpha\}}\int_0^x\frac{f\{t\}dt}{\{x-t\}^{\{\alpha-n+1\}}}, \quad (2.3) $$

$$ (0 < x, n = [Re(\alpha)] + 1) $$

and $$ (D_{-}^{\alpha,-\alpha,\eta}f)(x) = (D_{-}^{\alpha}f)(x) \equiv \left(\frac{-d}{dx}\right)^n\frac{1}{\Gamma(n-\alpha)}\int_x^{\infty}\frac{f(t)dt}{(x-t)^{(\alpha-n+1)}}, \quad (2.4) $$

$$(0 < x, n = [Re(\alpha)] + 1).$$

Again, if $\beta = 0$ the operators (1.1) and (1.2) reduce to as the Erdélyi-Kober fractional differential operators defined below [2, p. 109, Eqs. (2.6.35) (2.2.36)]:

$$\left(D_{0+}^{\alpha,0,\eta} f\right)(x) = \left(D_{\eta,\alpha}^{0+} f\right)(x) \equiv x^{-\eta} \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{t^{(\alpha+\eta)} f(t) dt}{(x-t)^{(\alpha-n+1)}}, \qquad (2.5)$$

$$(0 < x, n = [Re(\alpha)] + 1)$$

and $(D_{-}^{\alpha,0,\eta} f)(x) = \left(D_{\eta,\alpha}^{-} f\right)(x) \equiv x^{(\alpha+\eta)} \left(\frac{-d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_x^\infty \frac{t^{-\eta} f(t) dt}{(t-x)^{(\alpha-n+1)}},$ (2.6)

$$(0 < x, n = [Re(\alpha)]) .$$

The following lemma is required in the proof of our main results (see [4]):

Lemma 2.1 Let $\alpha, \beta, \eta \in \mathbb{C}. If\ \ 0 \le Re(\alpha), -min[0, Re(\alpha, \beta + \eta)] < Re(\sigma)$then

$$\left(D_{0+}^{\alpha,\beta,\eta} t^{\sigma-1}\right)(x) = x^{\sigma+\beta+1} \frac{\Gamma\sigma\ \Gamma(\sigma+\alpha+\beta+\eta)}{\Gamma(\sigma+\beta)\Gamma(\sigma+\eta)},\ \ x > 0 . \qquad (2.7)$$

In particular, for $x > 0$,

$$(D_{0+}^{\alpha,} t^{\sigma-1})(x) = x^{(\sigma-\beta-1)} \frac{\Gamma\sigma}{\Gamma(\sigma-\alpha)} \quad (0 \le Re(\alpha), Re(\sigma) > 0). \qquad (2.8)$$

$$\left(D_{\eta,\alpha}^{+} t^{\sigma-1}\right)(x) = x^{\sigma-1} \frac{\Gamma(\sigma+\alpha+\beta+\eta)}{\Gamma(\sigma+\eta)} \qquad (2.9)$$

$$\left(0 \le Re(\alpha), -Re(\alpha+\eta) < Re(\sigma)\right).$$

Lemma 2.2 Let $\alpha, \beta, \eta \in \mathbb{C}.\ \ If\ \ 0 \le Re(\alpha), (1 + min[Re(-\beta-\eta), Re(\alpha + \eta)]) > Re(\sigma),\ \ n = Re(\alpha) + 1$ then

$$\left(D_{-}^{\alpha,\beta,\eta} t^{\sigma-1}\right)(x) = x^{\sigma+\beta-1} \frac{\Gamma(1-\sigma-\beta)\Gamma(1-\sigma+\alpha+\eta)}{\Gamma(1-\sigma+\eta-\beta)\Gamma(1-\sigma)},\quad x > 0 \qquad (2.10)$$

In particular, for x > 0

$$(D_{-}^{\alpha,} t^{\sigma-1})(x) = x^{(\sigma+\alpha-1)} \frac{\Gamma(1-\sigma+\alpha)}{\Gamma(1-\sigma)}\ , \qquad (2.11)$$

$$[\ 0 \le Re(\alpha), (1 + Re(\alpha) - n) > Re(\sigma)] .$$

$$\left(D_{\eta,\alpha}^{-} t^{\sigma-1}\right)(x) = x^{(\sigma-1)} \frac{\Gamma(1-\sigma+\alpha+\eta)}{\Gamma(1-\sigma+\eta)}, \tag{2.12}$$

$$[\, 0 \leq Re(\alpha), [(1 + Re(\alpha+\eta) - n > Re(\sigma)] \,.$$

3. Main Results

In this section, we establish two Theorems involving the products of multivariable H-functions and the general class of polynomials associated with the Siago fractional differential operators. $[(1 + Re(\alpha+\eta) - n > Re(\sigma)]$.

Theorem 3.1 Let $\alpha, \beta, \delta, \rho, \sigma, \lambda_j, \eta_i, z_i, a, b, c_j \in \mathbb{C}$ with $\Re(\delta) > 0$ and $u_j > 0,\ v_i > 0,\ [\, i = (1, \dots, r);\ \ j = (1, \dots, s)]$. Then, we have

$$\{D_{0+}^{\alpha,\beta,\delta}(t^{\rho-1}(b-at)^{-\sigma} \prod_{j=1}^{s} {}_{p}^{\vartheta}M_{q}^{\mu}\,[c_j t^{u_j}(b-at)^{-\lambda_j}] \tag{3.1}$$

$$H[z_1 t^{v_1}(b-at)^{-\eta_1} \dots z_r t^{v_r}(b-at)^{-\eta_r}])\}(\mathrm{x})$$

$$= b^{-\sigma} x^{\rho+\beta-1} \sum_{k=o}^{\infty} \sum_{j=1}^{s} \frac{(a_1)_k, \dots, (a_p)_k}{(b_1)_k, \dots, (b_p)_k} \frac{x^{u_j k}}{\Gamma(\vartheta k+\mu)} (c_j)^k b^{-\sum_{j=1}^{s}\lambda_j k}\, x^{\sum_{j=1}^{s} u_j k}$$

$$H_{p+3,q+3:p_1,q_1;\dots;p_r,q_r;0,1}^{0,n+3:m_1,n_1;\dots;m_r,n_r;1,0}\left[\begin{matrix} z_1 \frac{x^{v_1}}{b^{\eta_1}} \\ \vdots \\ z_r \frac{x^{v_r}}{b^{\eta_r}} \\ -\frac{a}{b}x \end{matrix}\left|\begin{matrix} E, \left(a_j, \alpha_j^{(1)}, \dots, \alpha_j^{(r)}\right)_{1,p} : & \left(c_j^{(1)}, \gamma_j^{(1)}\right)_{1,p_1}; \dots; \left(c_j^{(r)}, \gamma_j^{(r)}\right)_{1,p_r}; \\ \left(b_j, \beta_j^{(1)}, \dots, \beta_j^{(r)}\right)_{1,q}, E^* : & \left(d_j^{(1)}, \delta_j^{(1)}\right)_{1,q_1}; \dots; \left(d_j^{(r)}, \delta_j^{(r)}\right)_{1,q_r}; (0,1) \end{matrix}\right.\right]$$

where E and E^* are defined simply by the following arrays:

$$E = \left(1-\sigma-\sum_{j=1}^{s}\lambda_j\, k\,; \eta_1, \dots, \eta_r, 1\right), \left(1-\rho-\sum_{j=1}^{s} u_j\, k\,; v_1, \dots, v_r, 1\right),$$

$$\left(1-\rho-\alpha-\beta-\delta-\sum_{j=1}^{s} u_j\, k\,; v_1, \dots, v_r, 1\right)$$

and $$E^* = \left(1-\sigma-\sum_{j=1}^{s}\lambda_j\, k\,; \eta_1, \dots, \eta_r, 0\right), \left(1-\rho-\sum_{j=1}^{s} u_j k\,; v_1, \dots, v_r, 1\right),$$

$$\left(1-\rho-\delta-\sum_{j=1}^{s} u_j\, k\,; v_1, \dots, v_r, 1\right).$$

The sufficient conditions of validity of (3.1) are,

$$|\arg\ z_i\,| < \frac{\pi}{2}\Omega_i\,,\quad where\ \ \Omega_i > 0,\ \forall i = 1, \dots, r\ and \tag{3.2}$$

$$\Omega_i = -\sum_{j=n+1}^{p} \alpha_j^{(i)} - \sum_{j=1}^{q} \beta_j^{(i)} + \sum_{j=1}^{n_i} \gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i} \gamma_j^{(i)} + \sum_{j=1}^{m_i} \delta_j^{(i)} - \sum_{j=m_i+1}^{q_i} \delta_j^{(i)}.$$

$$\Re(\mu) > 0, \quad (3.3)$$

$$\Re(\mu) + \sum_{i=1}^{r} \gamma_i \min_{1\le j\le m_i} \left\{ \Re\left(\frac{\left(d_j^{(i)}\right)}{\delta_j^{(i)}} \right) \right\} > max\{0, \Re(\beta-\delta)\},$$

$$\Re(\sigma) + \sum_{i=1}^{r} \eta_i \min_{1\le j\le m_i} \left\{ \Re\left(\frac{\left(d_j^{(i)}\right)}{\delta_j^{(i)}} \right) \right\} > max\{0, \Re(\beta-\delta)\}$$

and
$$\left|\frac{a}{b}x\right| < 1. \quad (3.4)$$

Proof: Let $\mathcal{L}$ be the left-hand side of (2.1). Using (1.4), (1.1), we get by and expension of the term of $(b-at)^{-\zeta}$, we get

$$(b-at)^{-\zeta} = b^{-\zeta} \sum_{s=0}^{\infty} \frac{(\zeta)_s}{s!} \left(\frac{ax}{b}\right)^s, \quad \text{where} \left(\frac{ax}{b} < 1\right). \quad (3.5)$$

Interchanging the summations and the integrals, after a little simplification, we obtain

$$\mathcal{L} = \sum_{k=o}^{\infty} \sum_{j=1}^{s} \frac{(a_1)_k, \dots, (a_p)_k}{(b_1)_k, \dots, (b_p)_k} \frac{x^{u_j k}}{\Gamma(\vartheta k + \mu)} (c_j)^k b^{-\sum_{j=1}^{s} \eta_j k}$$

$$\frac{1}{(2\pi i)^{r+1}} \int_{L_1} \dots \int_{L_r} \phi_1(\xi_1) \dots \phi_r(\xi_r) \psi(\xi_1, \dots, \xi_r) z_1^{\xi_1}, \dots, z_r^{\xi_r} d\xi_1 \dots d\xi_r$$

$$\int_{L_{r+1}} \frac{\Gamma\left(1-\sigma-\sum_{j=1}^{s} \lambda_j k + \sum_{i=1}^{r} \eta_i \xi_i + \xi_{r+1}\right)}{\Gamma\left(1-\sigma-\sum_{j=1}^{s} \lambda_j k + \sum_{i=1}^{r} \eta_i \xi_i\right) \Gamma(1+\xi_{r+1})} (-a/b)^{\xi_{r+1}} d\xi_{r+1}$$

$$\left(D_{0+}^{\alpha,\beta,\delta}\, t^{\rho+\sum_{j=1}^{s} \lambda_j k + \sum_{i=1}^{r} \eta_i \xi_i + \xi_{r+1} - 1}\right)(x). \quad (3.6)$$

Then applying (2.7) to the last integral in (3.6) and interpreting the involved Mellin-Barnes contour integrals in terms of the multivariable H-functions of r+1 variable, we obtain the right-hand side of (3.1).

If we put $\beta = -\alpha$ in **Theorem 3.1** and use (2.8), we arrive at the following new and interesting corollary concerning Riemann-Liouville fractional differential operators defined by (2.3) :

Corollary 3.1 $\{D_{0+}^{\alpha}(t^{\rho-1}(b-at)^{-\sigma}\prod_{j=1}^{s}{}_{p}^{\vartheta}M_{q}^{\mu}[c_j t^{u_j}(b-at)^{-\lambda_j}]$ (3.7)

$$H[z_1 t^{v_1}(b-at)^{-\eta_1}\dots z_r t^{v_r}(b-at)^{-\eta_r}])\}(x)$$

$$=b^{-\sigma}x^{\rho-\alpha-1}\sum_{k=o}^{\infty}\sum_{j=1}^{s}\frac{(a_1)_k,\dots,(a_p)_k}{(b_1)_k,\dots,(b_p)_k}\frac{x^{u_j k}}{\Gamma(\vartheta k+\mu)}(c_j)^k b^{-\sum_{j=1}^{s}\lambda_j k}x^{\sum_{j=1}^{s}u_j k}$$

$$H_{p+2,q+2:\,p_1,q_1;\dots;p_r,q_r;0,1}^{0,n+2:\,m_1,n_1;\dots;m_r,n_r;1,0}\left[\begin{matrix} z_1\frac{x^{v_1}}{b^{\eta_1}} \\ \vdots \\ z_r\frac{x^{v_r}}{b^{\eta_r}} \\ -\frac{a}{b}x\end{matrix}\left|\begin{matrix} E_1,\left(a_j,\alpha_j^{(1)},\dots,\alpha_j^{(r)}\right)_{1,p}: \left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,p_1};\dots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r}; \\ \left(b_j,\beta_j^{(1)},\dots,\beta_j^{(r)}\right)_{1,q},E_2: \left(d_j^{(1)},\delta_j^{(1)}\right)_{1,q_1};\dots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r};(0,1)\end{matrix}\right.\right].$$

$$E_1=\left(1-\sigma-\sum_{j=1}^{s}\lambda_j k;\eta_1,\dots,\eta_r,1\right),\left(1-\rho-\sum_{j=1}^{s}u_j k;v_1,\dots,v_r,1\right)$$

$$E_2=\left(1-\sigma-\sum_{j=1}^{s}\lambda_j k;\eta_1,\dots,\eta_r,0\right),\left(1-\rho+\alpha-\sum_{j=1}^{s}u_j k;v_1,\dots,v_r,1\right)$$

Again, if we choose $\beta=0$ in **Theorem 3.1** and use (2.4) we get the following result which is also believed to be new and pertains to Erdélyi -Kober fractional differential operators defined by (2.5):

Corollary 3.2 $\{D_{\delta,\alpha}^{+}(t^{\rho-1}(b-at)^{-\sigma}\prod_{j=1}^{s}{}_{p}^{\vartheta}M_{q}^{\mu}[c_j t^{u_j}(b-at)^{-\lambda_j}]$ (3.8)

$$H[z_1 t^{v_1}(b-at)^{-\eta_1}\dots z_r t^{v_r}(b-at)^{-\eta_r}])\}(x)$$

$$=b^{-\sigma}x^{\rho-1}\sum_{k=o}^{\infty}\sum_{j=1}^{s}\frac{(a_1)_k,\dots,(a_p)_k}{(b_1)_k,\dots,(b_p)_k}\frac{x^{u_j k}}{\Gamma(\vartheta k+\mu)}(c_j)^k b^{-\sum_{j=1}^{s}\lambda_j k}x^{\sum_{j=1}^{s}u_j k}$$

$$H_{p+2,q+2:\,p_1,q_1;\dots;p_r,q_r;0,1}^{0,n+2:\,m_1,n_1;\dots;m_r,n_r;1,0}\left[\begin{matrix} z_1\frac{x^{v_1}}{b^{\eta_1}} \\ \vdots \\ z_r\frac{x^{v_r}}{b^{\eta_r}} \\ -\frac{a}{b}x\end{matrix}\left|\begin{matrix} E_1^{*},\left(a_j,\alpha_j^{(1)},\dots,\alpha_j^{(r)}\right)_{1,p}: \left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,p_1};\dots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r};\_ \\ \left(b_j,\beta_j^{(1)},\dots,\beta_j^{(r)}\right)_{1,q},E_1^{**}: \left(d_j^{(1)},\delta_j^{(1)}\right)_{1,q_1};\dots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r};(0,1)\end{matrix}\right.\right]$$

$$E_1^{*}=\left(1-\sigma-\sum_{j=1}^{s}\lambda_j k;\eta_1,\dots,\eta_r,1\right),\left(1-\rho-\delta-\sum_{j=1}^{s}u_j k;v_1,\dots,v_r,1\right)$$

$$E_1^{**}=\left(1-\sigma-\sum_{j=1}^{s}\lambda_j k;\eta_1,\dots,\eta_r,0\right),\left(1-\rho+\alpha-\delta-\sum_{j=1}^{s}u_j k;v_1,\dots,v_r,1\right)$$

The sufficient conditions of validity of (3.7) are

$$0<\Re(\alpha) \quad (3.9)$$

$$\Re(\rho)+\sum_{i=1}^{r}\gamma_i \min_{1\le j\le m_i}\left\{\Re\left(\frac{\left(d_j^{(i)}\right)}{\delta_j^{(i)}}\right)\right\}>-R(\delta)\,,$$

$$\Re(\sigma)+\sum_{i=1}^{r}\lambda_i \min_{1\le j\le m_i}\left\{\Re\left(\frac{\left(d_j^{(i)}\right)}{\delta_j^{(i)}}\right)\right\}>-R(\delta)$$

and the conditions (3.2),(3.3) and (3.4) in **Theorem 3.1** are also satisfied.

***Theorem* 3.2** Let $\alpha,\beta,\delta,\rho,\sigma,\lambda_j,\eta_i,z_i,a,b,c_j\in\Re(\delta)>0$, and $u_j>0, v_i>0$, ($i\epsilon=\{1,\dots,r\}$; $j\epsilon=\{1,\dots,s\}$). Then we have

$$\{D_{-}^{\alpha,\beta,\delta}(t^{\rho-1}(b-at)^{-\sigma}\prod_{j=1}^{s}{}_{p}^{\vartheta}M_q^{\mu}\,[c_j t^{u_j}(b-at)^{-\lambda_j}] \tag{3.10}$$

$$H[z_1 t^{v_1}(b-at)^{-\eta_1}\dots z_r t^{v_r}(b-at)^{-\eta_r}])\}(\mathrm{x})$$

$$=b^{-\sigma}x^{\rho+\beta-1}\sum_{k=o}^{\infty}\sum_{j=1}^{s}\frac{(a_1)_k,\dots,(a_p)_k}{(b_1)_k,\dots,(b_p)_k}\frac{x^{u_jk}}{\Gamma(\vartheta k+\mu)}(c_j)^k b^{-\sum_{j=1}^{s}\lambda_j k}\,x^{\sum_{j=1}^{s}u_j k}$$

$$H_{p+3,q+3:\,p_1,q_1;\,\dots;\,p_r,q_r;0,1}^{0,n+3:\,m_1,n_1;\,\dots;\,m_r,n_r;1,0}\left[\begin{matrix} z_1\frac{x^{\gamma_1}}{b^{\eta_1}}\\ \vdots\\ z_r\frac{x^{\gamma_r}}{b^{\eta_r}}\\ -\frac{a}{b}x\end{matrix}\left|\begin{matrix}F,\left(a_j,\alpha_j^{(1)},\dots,\alpha_j^{(r)}\right)_{1,p}: & \left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,p_1};\dots; & \left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r};\\ \left(b_j,\beta_j^{(1)},\dots,\beta_j^{(r)}\right)_{1,q},F^*: & \left(d_j^{(1)},\delta_j^{(1)}\right)_{1,q_1};\dots; & \left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r};(0,1)\end{matrix}\right.\right]$$

where, for simplicity F & F^* are denoted by the following arrays:

$$F=\left(1-\sigma-\sum_{j=1}^{s}\lambda_j\,k\,;\eta_1,\dots,\eta_r,1\right),\left(\rho+\beta+\sum_{j=1}^{s}u_j\,k\,;v_1,\dots,v_r,1\right)$$

$$\left(\rho-\alpha-\delta-\sum_{j=1}^{s}u_j\,k\,;v_1,\dots,v_r,1\right).$$

$$F^*=\left(1-\sigma-\sum_{j=1}^{s}\lambda_j\,k\,;\eta_1,\dots,\eta_r,0\right),\left(\rho+\sum_{j=1}^{s}u_j\,k\,;v_1,\dots,v_r,1\right),$$

$$\left(\rho+\beta-\delta+\sum_{j=1}^{s}u_j\,k\,;v_1,\dots,v_r,1\right).$$

The sufficient conditions of validity of (3.7) are

$$0<\Re(\alpha), \tag{3.11}$$

$$\Re(\rho)+\sum_{i=1}^{r}\gamma_i \min_{1\le j\le m_i}\left\{\Re\left(\frac{\left(d_j^{(i)}\right)}{\delta_j^{(i)}}\right)\right\}>-R(\delta) \tag{3.12}$$

$$\Re(\sigma) + \sum_{i=1}^{r} \lambda_i \min_{1 \le j \le m_i} \left\{ \Re\left(\frac{\left(d_j^{(i)}\right)}{\delta_j^{(i)}} \right) \right\} > -R(\delta) \qquad (3.13)$$

and the condition (3.2), (3.3) and (3.4) in ***Theorem* 3.1** are also satisfied.

Proof: We easily obtain the **Theorem 3.2** after a small simplification on making use of similar lines as adopted in **Theorem 3.1** and using **Lemma 1.3.**

If we put $\beta = -\alpha$ in **Theorem 3.2,** and using (2.11),we arrive at the following new and interesting corollary concerning Riemann-Liouville fractional differential operators defined by (2.4) :

Corollary 3.5 $\{D_{-}^{\alpha}(t^{\rho-1}(b-at)^{-\sigma} \prod_{j=1}^{s} {}_{p}^{\vartheta}M_{q}^{\mu} [c_j t^{u_j}(b-at)^{-\lambda_j}]$ (3.14)

$$H[z_1 t^{v_1}(b-at)^{-\eta_1} \dots z_r t^{v_r}(b-at)^{-\eta_r}])\}(\mathrm{x})$$

$$= b^{-\sigma} x^{\rho-\alpha-1} \sum_{k=o}^{\infty} \sum_{j=1}^{s} \frac{(a_1)_k,\dots,(a_p)_k}{(b_1)_k,\dots,(b_p)_k} \frac{x^{u_j k}}{\Gamma(\vartheta k+\mu)} (c_j)^k b^{-\sum_{j=1}^{s} \eta_j k} x^{\sum_{j=1}^{s} u_j k}$$

$$H_{p+2,q+2:\,p_1,q_1;\,\dots\,;\,p_r,q_r;0,1}^{0,n+2:\,m_1,n_1;\,\dots\,;\,m_r,n_r;1,0} \left[\begin{matrix} z_1 \frac{x^{v_1}}{b^{\eta_1}} \\ \vdots \\ z_r \frac{x^{v_r}}{b^{\eta_r}} \\ -\frac{a}{b} x \end{matrix} \middle| \begin{matrix} E_2{}^{*}, \left(a_j, \alpha_j^{(1)}, \dots, \alpha_j^{(r)}\right)_{1,p} : & \left(c_j^{(1)}, \gamma_j^{(1)}\right)_{1,p_1} ; \dots ; & \left(c_j^{(r)}, \gamma_j^{(r)}\right)_{1,p_r} ; \\ \left(b_j, \beta_j^{(1)}, \dots, \beta_j^{(r)}\right)_{1,q}, E_2{}^{**} : & \left(d_j^{(1)}, \delta_j^{(1)}\right)_{1,q_1} ; \dots ; & \left(d_j^{(r)}, \delta_j^{(r)}\right)_{1,q_r} ; (0,1) \end{matrix} \right].$$

$$E_2{}^{*} = \left(1 - \sigma - \sum_{j=1}^{s} \lambda_j\, k; \eta_1, \dots, \eta_r, 1\right), \left(\beta + \rho + \sum_{j=1}^{s} u_j\, k; v_1, \dots, v_r, 1\right)$$

$$E_2{}^{**} = \left(1 - \sigma - \sum_{j=1}^{s} \lambda_j\, k\, ; \eta_1, \dots, \eta_r, 0\right), \left(\rho + \sum_{j=1}^{s} u_j\, k\, ; v_1, \dots, v_r, 1\right)$$

Again, if we $\beta = 0$ in **Theorem 3.2,** and use (2.12), we get the following result which is also believed to be new and pertains to Erdélyi-Kober fractional differential operators defined by (2.6) :

Corollary 3.4 $\{D_{\delta,\alpha}^{+}(t^{\rho-1}(b-at)^{-\sigma} \prod_{j=1}^{s} {}_{p}^{\vartheta}M_{q}^{\mu} \left[c_j t^{u_j}(b-at)^{-\lambda_j}\right]$ *(3.15)*

$$H[z_1 t^{v_1}(b-at)^{-\eta_1} \dots z_r t^{v_r}(b-at)^{-\eta_r}])\}(x)$$

$$=b^{-\sigma}x^{\rho-1}\sum_{k=o}^{\infty}\sum_{j=1}^{s}\frac{(a_1)_k,\ldots,(a_p)_k}{(b_1)_k,\ldots,(b_p)_k}\frac{x^{u_jk}}{\Gamma(\vartheta k+\mu)}(c_j)^k b^{-\sum_{j=1}^{s}\lambda_j k}\,x^{\sum_{j=1}^{s}u_jk}$$

$$H^{0,n+2:\,m_1,n_1;\,\ldots\,;\,m_r,n_r;1,0}_{p+2,q+2:\,p_1,q_1;\,\ldots\,;\,p_r,q_r;0,1}\left[\begin{matrix} z_1\frac{x^{v_1}}{b^{\eta_1}} \\ \vdots \\ z_r\frac{x^{v_r}}{b^{\eta_r}} \\ -\frac{a}{b}x \end{matrix}\left|\begin{matrix} F_1{}^*,\left(a_j,\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{1,p}:\ \left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,p_1};\ldots;\ \left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r}; \\ \left(b_j,\beta_j^{(1)},\ldots,\beta_j^{(r)}\right)_{1,q},F_1{}^{**}:\ \left(d_j^{(1)},\delta_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r};(0,1) \end{matrix}\right.\right]$$

$$F_1{}^* = \left(1-\sigma-\sum_{j=1}^{s}\lambda_j\,k;\eta_1,\ldots,\eta_r,1\right),\left(\rho-\alpha-\delta+\sum_{j=1}^{s}\mathrm{u}_j\,k\,;v_1,\ldots,v_r,1\right)$$

$$F_1{}^{**} = \left(1-\sigma-\sum_{j=1}^{s}\lambda_j\,k\,;\eta_1,\ldots,\eta_r,0\right),\left(\rho+\beta-\delta+\sum_{j=1}^{s}\mathrm{u}_j\,k\,;v_1,\ldots,v_r,1\right).$$

The conditions of validity of the above results follow easily from the conditions given with **Theorem 3.4.**

4. Special Cases and Applications

We begin by remarking the following facts:

(i) The generalized fractional integral operators used in **Theorem 3.1** and **3.2** are unified ones in nature.

(ii) The product of the general class of polynomials given in **Theorem 3.1** and **3.2** reduces to a large spectrum of known polynomials as illustrated in Srivastava et al [10].

(iii) The multivariable H-functions occurring in **Theorem 3.1** and **3.2** can be suitably specialized to give a large number of useful functions, for example, the Generalized Mittag-Leffler function, Bessel functions of one variable, generalized Wright hypergeometric functions, generalized Lauricella function, and so on.

(a) If we reduce the multivariable *H*-functions involved in (3.1) to the product of *r* different Whittaker functions and Srivastava et al. [10, p. 18, equation (2.6.7)] and taking, $v_i = 1,\ \eta_i \to 0$ (i = 1,…, r), we get the following new and interesting result :

$$\{D_{0+}^{\alpha,\beta,\delta}(t^{\rho-1}(b-at)^{-\sigma}\prod_{j=1}^{s}{}_p^{\vartheta}M_q^{\mu}\left[c_jt^{u_j}(b-at)^{-\lambda_j}\right]\times$$

$$\prod_{i=1}^{r}e^{\left(\frac{-tz_i}{2}\right)}W_{v_i,\rho_i}(tz_i)\}(\mathrm{x})$$

$$=b^{-\sigma}x^{\rho+\beta-1}\sum_{k=o}^{\infty}\sum_{j=1}^{s}\frac{(a_1)_k,\ldots,(a_p)_k}{(b_1)_k,\ldots,(b_p)_k}\frac{x^{u_jk}}{\Gamma(\vartheta k+\mu)}(c_j)^k b^{-\sum_{j=1}^{s}\eta_j k}\, x^{\sum_{j=1}^{s}u_j k}$$

$$H^{0,3:2,0\ ;\ldots;2,0;1,0}_{3,3:1,2;\ldots;1,2;0,1}\left[\begin{matrix} z_1 x\\ \vdots\\ z_r x\\ -\frac{a}{b}x\end{matrix}\left|\begin{matrix}(1-\sigma-\sum_{j=1}^{s}\lambda_j k\,;1,\ldots,1,1),(1-\rho-\sum_{j=1}^{s}u_j k\,;1,\ldots,1,1)\\ (1-\sigma-\sum_{j=1}^{s}\lambda_j k\,;1,\ldots,1,0)(1-\mu-\beta-\sum_{j=1}^{s}\lambda_j k\,;1,\ldots,1,1)\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(1-\rho-\alpha-\beta-\delta-\sum_{j=1}^{s}u_j k\,;1,\ldots,1,1):(1-u_1,1);\ldots;(1-u_r,1);-\\ (1-\rho-\delta-\sum_{j=1}^{s}\eta_j k\,;1,\ldots,1,1):\left(\frac{1}{2}\pm\rho_1,1\right);\ldots;\left(\frac{1}{2}\pm\rho_r,1\right);(0,1)\end{matrix}\right] \quad (4.1)$$

The conditions of validity of the (4.1) result easily follow the form (3.1).

(b) If we reduce the multivariable *H*-functions into the product of *H*-functions of two variables in **Theorem 3.1** and then reduce Fox's *H*-function to the exponential function by taking $v_i = 1,\ \eta_i \to 0$, we get the following result after a simple simplification which is believed to be new:

$$D^{\alpha,\beta,\delta}_{0+}\left(t^{\rho-1}(b-at)^{-\sigma}\prod_{j=1}^{s}{}^{\vartheta}_{p}M^{\mu}_{q}\left[c_j t^{u_j}(b-at)^{-\lambda_j}\right]e^{(-tz_1)}\times\right.$$

$$\left.H^{m_2,n_2}_{p_2,q_2}\left[z_2 t^{v_2}(b-at)^{-\eta_2}\left|\begin{matrix}\left(c_j^{(2)},\gamma_j^{(2)}\right)_{1,p_2}\\ \left(d_j^{(2)},\delta_j^{(2)}\right)_{1,q_2}\end{matrix}\right.\right]\right)(x)$$

$$=b^{-\sigma}x^{\rho+\beta-1}\sum_{k=o}^{\infty}\sum_{j=1}^{s}\frac{(a_1)_k,\ldots,(a_p)_k}{(b_1)_k,\ldots,(b_p)_k}\frac{x^{u_jk}}{\Gamma(\vartheta k+\mu)}(c_j)^k b^{-\sum_{j=1}^{s}\eta_j k}x^{\sum_{j=1}^{s}u_j k}$$

$$H^{0,3:1,0;m_2,n_2,\ 1,0}_{3,3:0,1\ ;p_2,q_2,\ 0,1}\left[\begin{matrix} z_1 x\\ z_2\frac{x^{v_2}}{b^{\eta_2}}\\ -\frac{a}{b}x\end{matrix}\left|\begin{matrix}(1-\sigma-\sum_{j=1}^{s}\lambda_j k;1,\eta_2,1),(1-\rho-\sum_{j=1}^{s}u_j k;1,v_2,1)\\ (1-\sigma-\sum_{j=1}^{s}\lambda_j k;1,\eta_2,0)(1-\mu-\beta-\sum_{j=1}^{s}\lambda_j k;1,v_2,1)\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(1-\rho-\alpha-\beta-\delta-\sum_{j=1}^{s}u_j k;1,v_2,1):-;\ \left(c_j^{(2)},\gamma_j^{(2)}\right)_{1,p_2};-\\ (1-\rho-\delta-\sum_{j=1}^{s}u_j k;1,v_2,1):(0,1);\ldots;\ \left(d_j^{(2)},\delta_j^{(2)}\right)_{1,q_2};(0,1)\end{matrix}\right]. \quad (4.2)$$

The conditions of validity of the (4.1) result easily follow from (3.1).

5. Conclusion

In the present paper, we have obtained the result namely **Theorem 3.1** and Theorem **3.2** of the generalized fractional derivative operators given by Saigö [18]. The theorems have been developed in terms of the product of multivariable *H*-functions [8,12]and a general class of polynomials in a compact and elegant with the help of Saigö operator. Most of the results obtained have been put in a compact form, avoiding the occurrence of [18] infinite series and thus making them useful in applications.

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