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FIXED POINT RESULTS TO STABILITIES OF FUNCTIONAL EQUATION IN NEUTROSOPHIC BANACH SPACES

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Abstract

In this article, we discuss various functional equations whose Hyer-Ulams-Rassiss stability has been investigated for a wide range of mathematical structures. Neutrosophic normed spaces serves as our discussion's frame work. Both Archimedean and non-Archimedean variations of these spaces are taken into consideration. An extension of the Banach contraction mapping principle on generalized metric spaces with permissible infinite distances is applied to get our results in this fixed point approach to the current challenge.

Keywords: *Stability, Banach space, Non-Archimidean, Contraction map, Functional equation.*

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1. Introduction

This study aim to develop certain stability conclusions for functional equation in non-Archimedean and Archimedean Neutrosophic Banach space. The result in this case indicate Hyer-Ulam-Rassias stability as the type of stability. Such stabilities have their roots in the work of Hyer, Ulam and Rassias, where Ulam formulated this problem for group homomorphism which types solved for Cauchy functional equations, and then Rassias expanded it to the case of linear mapping. This idea of stability has a fairly broad range of applications, including issues with differential equations, isometrics and other topics, it is commonly known that fuzzy concepts are current mathematical tenets that have gained traction In practically all areas of mathematical study. Essentially, we use a fixed point theorem to generalised metric spaces to get our key conclusions. It should also be noted that various areas of mathematics are covered by the study of Hyer-Ulam-Rassias stability. The literature on this topic is very extensive.

2. Preliminaries

Definition 2.1. [14] Let $\mathcal{V}$ be a linear space over a field $\mathcal{K}$ with a non-Archimedean valuation $|.|$. A mapping $||.|| : \mathcal{V} \to [0,\infty)$ is said to be a non-Archimedean norm if it holds the following assertions:

(i) $||\mathfrak{v}|| = 0 \Leftrightarrow \mathfrak{v} = 0$;

(ii) $||\mathfrak{k}\mathfrak{v}||=|\mathfrak{k}|||\mathfrak{v}||$;

(iii) The strong triangle inequality $||\mathfrak{v}+\mathfrak{w}|| \le \max\{||\mathfrak{v}||, ||\mathfrak{w}||\}$ for all $, \mathfrak{w} \in \mathcal{V}$.

Then $(\mathcal{V}, ||.||)$ is called a non-Archimedean normed space [*NANS*].

Definition 2.2. [1] Let $\mathfrak{L}$ be any nonempty set. An intuitionistic fuzzy set $\mathcal{S}$ of $\mathfrak{L}$ is an object of the form $\mathcal{S} = \{(\mathfrak{v}, \eta_{\mathcal{S}}(\mathfrak{v}), \upsilon_{\mathcal{S}}(\mathfrak{v}))/\ \mathfrak{v} \in \mathfrak{L}\}$, where the functions $\eta_{\mathcal{S}} : \mathfrak{L} \to [0,1]$ and $\upsilon_{\mathcal{S}} : \mathfrak{L} \to [0,1]$ denote the degree of nearness and degree of non-nearness of the element $\mathfrak{v} \in \mathfrak{L}$, $0 \le \eta_{\mathcal{S}}(\mathfrak{v}) + \upsilon_{\mathcal{S}}(\mathfrak{v}) \le 1$.

Definition 2.3. The 7-tuple $(\mathcal{V}, \eta, \upsilon, \varsigma, \star, \diamond)$ is said to be a Non-Archimedean Neutrosophic Normed Space [*NANNS*], if $\mathcal{V}$ is a vector space over a field $\mathbb{R}$, $\star$ is a continuous t-norm, $\diamond$ is a continous t-conorm, and $\eta, \upsilon, \varsigma$ are functions from $\mathcal{V} \times \mathbb{R} \to [0,1]$ meets the following conditions for every $\mathfrak{v}, \mathfrak{w}, \mathfrak{z} \in \mathcal{V}$ and $\sigma, \tau \in \mathbb{R}$

(n1) $0 \leq \eta(\mathfrak{v},\tau) \leq 1;\ 0 \leq \upsilon(\mathfrak{v},\tau) \leq 1;\ 0 \leq \varsigma(\mathfrak{v},\tau) \leq 1;$

(n2) $\eta(\mathfrak{v},\tau) + \upsilon(\mathfrak{v},\tau) + \rho(\mathfrak{v},\tau) \leq 3;$

(n3) $\eta(\mathfrak{v},\tau) > 0;$

(n4) $\eta(\mathfrak{v},\tau) = 1 \Leftrightarrow \mathfrak{v} = \tau;$

(n5) $\eta(\sigma\mathfrak{v},\tau) = \eta\left(\mathfrak{v},\frac{\tau}{|\sigma|}\right)$ for $\sigma \neq 0;$

(n6) $\eta(\mathfrak{v},\sigma) \star \eta(\mathfrak{w},\tau) \leq \eta(\mathfrak{v}+\mathfrak{w},\ \max\{\sigma,\tau\});$

(n7) $\eta(\mathfrak{v},\tau) : (0,\infty) \to [0,1]$ is continuous;

(n8) $\lim_{\tau\to\infty} \eta(\mathfrak{v},\tau) = 1$ and $\lim_{\tau\to 0} \eta(\mathfrak{v},\tau) = 0;$

(n9) $\upsilon(\mathfrak{v},\tau) < 1;$

(n10) $\upsilon(\mathfrak{v},\tau) = 0 \Leftrightarrow \mathfrak{v} = 0;$

(n11) $\upsilon(\sigma\mathfrak{v},\tau) = \upsilon\left(\mathfrak{v},\frac{\tau}{|\sigma|}\right)$ for $\sigma \neq 0;$

(n12) $\upsilon(\mathfrak{v},\sigma) \diamond \upsilon(\mathfrak{w},\tau) \geq \upsilon(\mathfrak{v}+\mathfrak{w},\ \max\{\sigma,\tau\});$

(n13) $\upsilon(\mathfrak{v},\tau) : (0,\infty) \to [0,1]$ is continuous;

(n14) $\lim_{\tau\to\infty} \upsilon(\mathfrak{v},\tau) = 0$ and $\lim_{\tau\to 0} \upsilon(\mathfrak{v},\tau) = 1;$

(n15) $\varsigma(\mathfrak{v},\tau) < 1;$

(n16) $\varsigma(\mathfrak{v},\tau) = 0 \Leftrightarrow \mathfrak{v} = 0;$

(n17) $\varsigma(\sigma\mathfrak{v},\tau) = \varsigma\left(\mathfrak{v},\frac{\tau}{|\sigma|}\right)$ for $\sigma \neq 0;$

(n18) $\varsigma(\mathfrak{v},\sigma) \diamond \varsigma(\mathfrak{w},\tau) \geq \varsigma(\mathfrak{v}+\mathfrak{w},\ \max\{\sigma,\tau\});$

(n19) $\varsigma(\mathfrak{v},\tau) : (0,\infty) \to [0,1]$ is continuous;

(n20) $\lim_{\tau\to\infty} \varsigma(\mathfrak{v},\tau) = 0$ and $\lim_{\tau\to 0} \varsigma(\mathfrak{v},\tau) = 1.$

Remark 2.4. From (n4) and (n6), it follows that) $\eta(\mathfrak{v},\tau)$ is non-decreasing function on $(0,\infty)$ and from (n10),(n12) and (n16),(n18), it follows that $\upsilon(\mathfrak{v},\tau)$ and $\varsigma(\mathfrak{v},\tau)$ are non-increasing function on $(0,\infty)$.

Example 2.5. Let $(\mathcal{V},\|.\|)$ be a non-Archimedean normed space [*NANS*], and let $\mathfrak{r} \star \mathfrak{s} = \mathfrak{r}\mathfrak{s}$, $\mathfrak{r} \diamond \mathfrak{s} = \min\{\mathfrak{r},\mathfrak{s}\}$ for all $\mathfrak{r},\mathfrak{s} \in [0,1]$. Let $\eta(\mathfrak{v},\tau) = \frac{\tau}{\tau+\|\mathfrak{v}\|}$, $\upsilon(\mathfrak{v},\tau) = \frac{\|\mathfrak{v}\|}{\tau+\|\mathfrak{v}\|}$ and $\varsigma(\mathfrak{v},\tau) = \frac{\|\mathfrak{v}\|}{\tau}$ for all $\mathfrak{v} \in \mathcal{V}$. Then $(\mathcal{V},\eta,\upsilon,\varsigma,\star,\diamond)$ is *NANNS*.

Definition 2.6. Let $(\mathcal{V},\eta,\upsilon,\rho,\star,\diamond)$ be a *NANNS*. A sequence $\{\mathfrak{v}_n\}$ in $\mathcal{V}$ is said to be converge if there exist an $\mathfrak{v}\in\mathcal{V}$ for all $\tau>0$, such that $\lim_{n\to\infty}\eta(\mathfrak{v}_n-\mathfrak{v},\tau)=1$, $\lim_{n\to\infty}\upsilon(\mathfrak{v}_n-\mathfrak{v},\tau)=0$ and $\lim_{n\to\infty}\rho(\mathfrak{v}_n-\mathfrak{v},\tau)=0$. In this case, $\mathfrak{v}$ is called the limit of sequence $\{\mathfrak{v}_n\}$ and it is indicated by $(\eta,\upsilon,\rho)-\lim_{n\to\infty}\mathfrak{v}_n=\mathfrak{v}$.

Definition 2.7. Let $(\mathcal{V},\eta,\upsilon,\varsigma,\star,\diamond)$ be a *NANNS*. A sequence $\{\mathfrak{v}_n\}$ in $\mathcal{V}$ is named a Cauchy sequence if for each $\varepsilon>0$ and $\tau>0$ there exists an $m\in\mathbb{N}$ such that for all $n\ge m$ and $q>0$, we have $\eta\left(\mathfrak{v}_{n+q}-\mathfrak{v}_n,\tau\right)>1-\varepsilon$, $\upsilon\left(\mathfrak{v}_{n+q}-\mathfrak{v}_n,\tau\right)<\varepsilon$ and $\varsigma\left(\mathfrak{v}_{n+q}-\mathfrak{v}_n,\tau\right)<\varepsilon$.

Definition 2.8. Let $(\mathcal{V},\eta,\upsilon,\varsigma,\star,\diamond)$ be a *NANNS*. Then $(\mathcal{V},\eta,\upsilon,\varsigma,\star,\diamond)$ is said to be complete if every Cauchy sequence is convergent. In this case $(\mathcal{V},\eta,\upsilon,\varsigma,\star,\diamond)$ is named a non-Archimedean neutrosophic Banach space.

Theorem 2.9.[3] Let $(\mathcal{V},\mathfrak{d})$ be a complete generalized metric space and let $\mathfrak{T}:\mathcal{V}\to\mathcal{V}$ be a strictly contractive mapping with Lipschitz constant $0<\mathcal{T}<1$, that is, $\mathfrak{d}(\mathfrak{T}\mathfrak{v},\mathfrak{T}\mathfrak{w})\le\mathcal{T}\mathfrak{d}(\mathfrak{v},\mathfrak{w})$, for all $\mathfrak{v},\mathfrak{w}\in\mathcal{V}$. Then for each $\mathfrak{v}\in\mathcal{V}$, either $\mathfrak{d}(\mathfrak{T}^n\mathfrak{v},\mathfrak{T}^{n+1}\mathfrak{v})<\infty$ for all $n\ge m_0$ for some non-negative integer m_0. Moreover, if the second alternative holds then

(1) the sequence $\{\mathfrak{T}^n\mathfrak{v}\}$ converges to a fixed point $\mathfrak{w}^*$ of $\mathfrak{T}$;

(2) $\mathfrak{w}^*$ is the unique fixed point of $\mathfrak{T}$ in the set $\mathcal{W}=\{\mathfrak{w}\in\mathcal{V}:\mathfrak{d}(\mathfrak{T}^{m_0}\mathfrak{v},\mathfrak{w})<\infty\}$;

(3) $\mathfrak{d}(\mathfrak{w},\mathfrak{w}^*)\le(\frac{1}{1-\mathcal{T}})\mathfrak{d}(\mathfrak{w},\mathfrak{T}\mathfrak{w})$ for all $\mathfrak{w}\in\mathcal{V}$.

Lemma 2.10. [18] Let $\mathcal{V}$ and $\mathcal{W}$ be linear spaces. If a mapping $\xi:\mathcal{V}\to\mathcal{W}$ satisfies $\xi(\mathfrak{v}+\mathfrak{w})-\xi(\mathfrak{v})-\xi(\mathfrak{w})=\rho(2\xi\left(\frac{\mathfrak{v},+\mathfrak{w}}{2}\right)-\xi(\mathfrak{v})-\xi(\mathfrak{w}))$ for all $\mathfrak{v},\mathfrak{w}\in,\mathcal{V}$ then $\xi:\mathcal{V}\to\mathcal{W}$ is additive.

Lemma 2.11. [18] Let $\mathcal{V}$ and $\mathcal{W}$ be linear spaces. If a mapping $\xi:\mathcal{V}\to\mathcal{W}$ satisfies

$\xi(0) = 0$ and $\rho(\xi(\mathfrak{v},\mathfrak{w}) - \xi(\mathfrak{v}) - \xi(\mathfrak{w})) = 2\xi\left(\frac{\mathfrak{v},+\mathfrak{w}}{2}\right) - \xi(\mathfrak{v}) - \xi(\mathfrak{w})$ for all $\mathfrak{v},\mathfrak{w} \in \mathcal{V}$, then $\xi: \mathcal{V} \to \mathcal{W}$ is additive. For our notational reason we consider the $\xi: \mathcal{V} \to \mathcal{W}$ and define

$$\mathcal{D}_1\xi(\mathfrak{v},\mathfrak{w}) = \xi(\mathfrak{v} + \mathfrak{w}) - \xi(\mathfrak{v}) - \xi(\mathfrak{w}) - \rho(2\xi\left(\frac{\mathfrak{v},+\mathfrak{w}}{2}\right) - \xi(\mathfrak{v}) - \xi(\mathfrak{w})) \qquad \text{(I)}$$

$$\mathcal{D}_1\xi(\mathfrak{v},\mathfrak{w}) = (2\xi\left(\frac{\mathfrak{v},+\mathfrak{w}}{2}\right) - \xi(\mathfrak{v}) - \xi(\mathfrak{w})) - \rho(\xi(\mathfrak{v} + \mathfrak{w}) - \xi(\mathfrak{v}) - \xi(\mathfrak{w})) \qquad \text{(II)}$$

3. Non-Archimedean Neutrosophic Stability of Additive $\rho -$ Functional Equation (I)

Lemma 3.1. Let $(\mathfrak{Z},\eta',\upsilon',\varsigma')$ be a *NANNS* and $\Psi : \mathcal{V} \times \mathcal{V} \to \mathfrak{Z}$ be a function. Let $\mathfrak{L} = \{\wp : \mathcal{V} \to \mathcal{W};\ \wp(0) = 0\}$ and defined by

$$\mathfrak{d}(\wp,\mathscr{g}) = \inf\left\{\kappa \in \mathbb{R}^+ : \begin{cases}\eta(\wp(\mathfrak{v}) - \mathscr{g}(\mathfrak{v}),\kappa\tau) \geq \eta'(\Psi(\mathfrak{v},\mathfrak{v}),\tau),\\ \upsilon(\wp(\mathfrak{v}) - \mathscr{g}(\mathfrak{v}),\kappa\tau) \leq \upsilon'(\Psi(\mathfrak{v},\mathfrak{v}),\tau),\\ \varsigma(\wp(\mathfrak{v}) - \mathscr{g}(\mathfrak{v}),\kappa\tau) \leq \varsigma'(\Psi(\mathfrak{v},\mathfrak{v}),\tau),\end{cases} \ for\ all\ \mathfrak{v} \in \mathcal{V},\tau > 0\right\}$$

and $\wp,\mathscr{g} \in \mathfrak{L}$. Then $(\mathfrak{L},\mathfrak{d})$ is a complete generalized metric space.

Proof: Let $\xi,\wp,\mathscr{g} \in \mathfrak{L}$ and $\mathfrak{d}(\xi,\wp) = \kappa_1 < \infty$, $\mathfrak{d}(\wp,\mathscr{g}) = \kappa_2 < \infty$.

Then, $\begin{cases}\eta(\xi(\mathfrak{v}) - \wp(\mathfrak{v}),\kappa_1\tau) \geq \eta'(\Psi(\mathfrak{v},\mathfrak{v}),\tau)\\ \upsilon(\xi(\mathfrak{v}) - \wp(\mathfrak{v}),\kappa_1\tau) \leq \upsilon'(\Psi(\mathfrak{v},\mathfrak{v}),\tau)\\ \varsigma(\xi(\mathfrak{v}) - \wp(\mathfrak{v}),\kappa_1\tau) \leq \varsigma'(\Psi(\mathfrak{v},\mathfrak{v}),\tau).\end{cases}$ and

$$\begin{cases}\eta(\wp(\mathfrak{v}) - \mathscr{g}(\mathfrak{v}),\kappa_1\tau) \geq \eta'(\Psi(\mathfrak{v},\mathfrak{v}),\tau)\\ \upsilon(\wp(\mathfrak{v}) - \mathscr{g}(\mathfrak{v}),\kappa_1\tau) \leq \upsilon'(\Psi(\mathfrak{v},\mathfrak{v}),\tau)\\ \varsigma(\wp(\mathfrak{v}) - \mathscr{g}(\mathfrak{v}),\kappa_1\tau) \leq \varsigma'(\Psi(\mathfrak{v},\mathfrak{v}),\tau).\end{cases}$$

Therefore

$$\begin{cases}\eta(\xi(\mathfrak{v}) - \mathscr{g}(\mathfrak{v}),\max\{\kappa_1,\kappa_2\}\tau) \geq \eta(\xi(\mathfrak{v}) - \wp(\mathfrak{v}),\kappa_1\tau) \star \eta(\wp(\mathfrak{v}) - \mathscr{g}(\mathfrak{v}),\kappa_2\tau)\\ \qquad \geq \eta'(\Psi(\mathfrak{v},\mathfrak{v}),\tau) \star \eta'(\Psi(\mathfrak{v},\mathfrak{v}),\tau) \geq \eta'(\Psi(\mathfrak{v},\mathfrak{v}),\tau),\\ \upsilon(\xi(\mathfrak{v}) - \mathscr{g}(\mathfrak{v}),\max\{\kappa_1,\kappa_2\}\tau) \leq \upsilon(\xi(\mathfrak{v}) - \wp(\mathfrak{v}),\kappa_1\tau) \diamond \upsilon(\wp(\mathfrak{v}) - \mathscr{g}(\mathfrak{v}),\kappa_2\tau)\\ \qquad \leq \upsilon'(\Psi(\mathfrak{v},\mathfrak{v}),\tau) \diamond \upsilon'(\Psi(\mathfrak{v},\mathfrak{v}),\tau) \leq \upsilon'(\Psi(\mathfrak{v},\mathfrak{v}),\tau),\\ and\quad \varsigma(\xi(\mathfrak{v}) - \mathscr{g}(\mathfrak{v}),\max\{\kappa_1,\kappa_2\}\tau) \leq \varsigma(\xi(\mathfrak{v}) - \wp(\mathfrak{v}),\kappa_1\tau) \diamond \varsigma(\wp(\mathfrak{v}) - \mathscr{g}(\mathfrak{v}),\kappa_2\tau)\\ \qquad \leq \varsigma'(\Psi(\mathfrak{v},\mathfrak{v}),\tau) \circledast \varsigma'(\Psi(\mathfrak{v},\mathfrak{v}),\tau) \leq \varsigma'(\Psi(\mathfrak{v},\mathfrak{v}),\tau)\end{cases}$$

for all $\mathfrak{v} \in \mathcal{V}$, $\tau > 0$. Hence $\mathfrak{d}(\xi,\mathscr{g}) \leq \max\{\kappa_1,\kappa_2\}$ so that $\mathfrak{d}(\xi,\mathscr{g}) \leq \max\{\mathfrak{d}(\xi,\wp),\mathfrak{d}(\wp,\mathscr{g})\}$.

This proves the triangle inequality for $(\mathfrak{L}, \mathfrak{d})$ is a generalized metric space.

Now we prove that $(\mathfrak{L}, \mathfrak{d})$ is complete. Let $\{\wp_n\}$ be a Cauchy sequence in $(\mathfrak{L}, \mathfrak{d})$.

Now for each fixed $\mathfrak{v} \in \mathcal{V}$ and for every $\varepsilon > 0$ there exists $\varpi > 0$ such that

$$\begin{cases} \eta'\left(\Psi(\mathfrak{v},\mathfrak{v}), \frac{\tau}{\varpi}\right) > 1-\varepsilon, \\ \upsilon'\left(\Psi(\mathfrak{v},\mathfrak{v}), \frac{\tau}{\varpi}\right) < \varepsilon, \\ \varsigma'\left(\Psi(\mathfrak{v},\mathfrak{v}), \frac{\tau}{\varpi}\right) < \varepsilon. \end{cases}$$

Since $\{\wp_n\}$ is a Cauchy sequence in $(\mathfrak{L}, \mathfrak{d})$ corresponding to $\varpi > 0$, there exists $m_0 \in \mathbb{N}$ such that $\mathfrak{d}(\wp_n, \wp_m) < \varpi$ for all $m, n \geq m_0$.Since,

$$\mathfrak{d}(\wp_n, \wp_m) = \inf\left\{\kappa \in \mathbb{R}^+ \begin{cases} \eta(\wp_n(\mathfrak{v}) - \wp_m(\mathfrak{v}), \kappa\tau) \geq \eta'(\Psi(\mathfrak{v},\mathfrak{v}), \tau) \\ \upsilon(\wp_n(\mathfrak{v}) - \wp_m(\mathfrak{v}), \kappa\tau) \leq \upsilon'(\Psi(\mathfrak{v},\mathfrak{v}), \tau) \\ \varsigma(\wp_n(\mathfrak{v}) - \wp_m(\mathfrak{v}), \kappa\tau) \leq \varsigma'(\Psi(\mathfrak{v},\mathfrak{v}), \tau) \end{cases} for\ all\ \mathfrak{v} \in > 0 \right\}.$$

That is,

$$\mathfrak{d}(\wp_n, \wp_m) = \inf\left\{\kappa \in \mathbb{R}^+ \begin{cases} \eta(\wp_n(\mathfrak{v}) - \wp_m(\mathfrak{v}), \kappa\tau) \geq \eta'(\Psi(\mathfrak{v},\mathfrak{v}), \frac{\tau}{\kappa}) \\ \upsilon(\wp_n(\mathfrak{v}) - \wp_m(\mathfrak{v}), \kappa\tau) \leq \upsilon'\left(\Psi(\mathfrak{v},\mathfrak{v}), \frac{\tau}{\kappa}\right) \\ \varsigma(\wp_n(\mathfrak{v}) - \wp_m(\mathfrak{v}), \kappa\tau) \leq \varsigma'\left(\Psi(\mathfrak{v},\mathfrak{v}), \frac{\tau}{\kappa}\right) \end{cases} for\ all\ \mathfrak{v} \in > 0 \right\}$$

then there exists $\kappa_3 \in [0, \infty)$ such that $\mathfrak{d}(\wp_n, \wp_m) \leq \kappa_3 < \varpi$ for all $m, n \geq m_0$;

$\eta(\wp_n(\mathfrak{v}) - \wp_m(\mathfrak{v}), \kappa\tau) \geq \eta'\left(\Psi(\mathfrak{v},\mathfrak{v}), \frac{\tau}{\kappa_3}\right) \geq \eta'\left(\Psi(\mathfrak{v},\mathfrak{v}), \frac{\tau}{\varpi}\right) > 1-\in,$

$\upsilon(\wp_n(\mathfrak{v}) - \wp_m(\mathfrak{v}), \kappa\tau) \leq \upsilon'\left(\Psi(\mathfrak{v},\mathfrak{v}), \frac{\tau}{\kappa_3}\right) \leq \upsilon'\left(\Psi(\mathfrak{v},\mathfrak{v}), \frac{\tau}{\varpi}\right) \leq \in,$

and $\varsigma(\wp_n(\mathfrak{v}) - \wp_m(\mathfrak{v}), \kappa\tau) \leq \varsigma'\left(\Psi(\mathfrak{v},\mathfrak{v}), \frac{\tau}{\kappa_3}\right) \leq \varsigma'\left(\Psi(\mathfrak{v},\mathfrak{v}), \frac{\tau}{\varpi}\right) \leq \in.$

Since $\eta(\mathfrak{v}, \tau)$ is non-decreasing, $\upsilon(\mathfrak{v}, \tau)$ and $\varsigma(\mathfrak{v}, \tau)$ are non-increasing with respect to τ for all $m, n \geq m_0$. This show that for fixed $\mathfrak{v} \in \mathcal{V}$, $\{\wp_n(\mathfrak{v})\}$ is a Cauchy sequence in $\mathcal{W}$. Also since $\mathcal{W}$ is Banach space, for each fixed $\mathfrak{v} \in \mathcal{V}$, there exists $\wp(\mathfrak{v}) \in \mathcal{W}$ such that

$\wp(\mathfrak{v}) = (\eta, \upsilon, \varsigma) - \lim_{n\to\infty} \wp_n(\mathfrak{v})$. So, we have a mapping $\wp : \mathcal{V} \to \mathcal{W}$ such that

$\wp(\mathfrak{v}) = (\eta, \upsilon, \varsigma) - \lim_{n\to\infty} \wp_n(\mathfrak{v})$ for all $\mathfrak{v} \in \mathcal{V}$. Again $\{\wp_n\}$ is a Cauchy sequence in $(\mathfrak{L}, \mathfrak{d})$ so $\varepsilon > 0$ and $\tau > 0$ there exists $m_0 \in \mathbb{N}$ such that $\mathfrak{d}(\wp_n, \wp_m) < \kappa'$ for all $m, n \geq m_0$.

$$\begin{cases} \eta(\wp_n(\mathfrak{v}) - \wp_m(\mathfrak{v}), \kappa\tau) \geq \eta'\left(\Psi(\mathfrak{v},\mathfrak{v}), \frac{\tau}{\kappa'}\right) \geq \eta'\left(\Psi(\mathfrak{v},\mathfrak{v}), \frac{\tau}{\varepsilon}\right) > 1 - \varepsilon, \\ \upsilon(\wp_n(\mathfrak{v}) - \wp_m(\mathfrak{v}), \kappa\tau) \leq \upsilon'\left(\Psi(\mathfrak{v},\mathfrak{v}), \frac{\tau}{\kappa'}\right) \leq \upsilon'\left(\Psi(\mathfrak{v},\mathfrak{v}), \frac{\tau}{\varepsilon}\right) \leq \varepsilon, \\ \varsigma(\wp_n(\mathfrak{v}) - \wp_m(\mathfrak{v}), \kappa\tau) \leq \varsigma'\left(\Psi(\mathfrak{v},\mathfrak{v}), \frac{\tau}{\kappa'}\right) \leq \varsigma'\left(\Psi(\mathfrak{v},\mathfrak{v}), \frac{\tau}{\varepsilon}\right) \leq \varepsilon. \end{cases}$$

$$\Rightarrow \quad \begin{cases} \eta(\wp_n(\mathfrak{v}) - \wp_m(\mathfrak{v}), \varepsilon\tau) \geq \eta'(\Psi(\mathfrak{v},\mathfrak{v}), \tau), \\ \upsilon(\wp_n(\mathfrak{v}) - \wp_m(\mathfrak{v}), \varepsilon\tau) \leq \upsilon'(\Psi(\mathfrak{v},\mathfrak{v}), \tau), \\ \varsigma(\wp_n(\mathfrak{v}) - \wp_m(\mathfrak{v}), \varepsilon\tau) \leq \varsigma'(\Psi(\mathfrak{v},\mathfrak{v}), \tau). \end{cases}$$

Now, let $\varepsilon, \varrho > 0$ be given and $m, n > m_0$, $\tau > 0$, then

$$\begin{aligned} \eta(\wp_n(\mathfrak{v}) - \wp(\mathfrak{v}), \max\{\varepsilon,\varrho\}\tau) &\geq \eta(\wp_n(\mathfrak{v}) - \wp_m(\mathfrak{v}), \varepsilon\tau) \star \eta(\wp_m(\mathfrak{v}) - \wp(\mathfrak{v}), \tau\varrho) \\ &\geq \eta'(\Psi(\mathfrak{v},\mathfrak{v}), \tau) \star \eta(\wp_m(\mathfrak{v}) - \wp(\mathfrak{v}), \tau\varrho) \\ &\geq \eta'(\Psi(\mathfrak{v},\mathfrak{v}), \tau) \star 1, \text{ as } m \to \infty \\ &= \eta'(\Psi(\mathfrak{v},\mathfrak{v}), \tau) \text{ (by boundary condition),} \end{aligned}$$

and $\upsilon(\wp_n(\mathfrak{v}) - \wp(\mathfrak{v}), \max\{\varepsilon,\varrho\}\tau) \leq \upsilon(\wp_n(\mathfrak{v}) - \wp_m(\mathfrak{v}), \varepsilon\tau) \diamond \upsilon(\wp_m(\mathfrak{v}) - \wp(\mathfrak{v}), \tau\varrho)$

$$\begin{aligned} &\leq \upsilon'(\Psi(\mathfrak{v},\mathfrak{v}), \tau) \diamond \upsilon(\wp_m(\mathfrak{v}) - \wp(\mathfrak{v}), \tau\varrho) \\ &\leq \upsilon'(\Psi(\mathfrak{v},\mathfrak{v}), \tau) \diamond 0, \text{ as } m \to \infty \\ &= \upsilon'(\Psi(\mathfrak{v},\mathfrak{v}), \tau) \text{ (by boundary condition).} \end{aligned}$$

In addition, $\varsigma(\wp_n(\mathfrak{v}) - \wp(\mathfrak{v}), \max\{\varepsilon,\varrho\}\tau)$

$$\begin{aligned} &\leq \varsigma(\wp_n(\mathfrak{v}) - \wp_m(\mathfrak{v}), \varepsilon\tau) \diamond \varsigma(\wp_m(\mathfrak{v}) - \wp(\mathfrak{v}), \tau\varrho) \\ &\leq \varsigma'(\Psi(\mathfrak{v},\mathfrak{v}), \tau) \diamond \varsigma(\wp_m(\mathfrak{v}) - \wp(\mathfrak{v}), \tau\varrho) \\ &\leq \varsigma'(\Psi(\mathfrak{v},\mathfrak{v}), \tau) \diamond 0, \text{ as } m \to \infty \\ &= \varsigma'(\Psi(\mathfrak{v},\mathfrak{v}), \tau) \text{ (by boundary condition).} \end{aligned}$$

That is, $\mathfrak{d}(\wp_n, \wp) \leq \max\{\varepsilon, \varrho\}$ for all $\mathfrak{v} \in \mathcal{V}$ and $m, n \geq m_0$.

Taking $\varrho \to \infty$ we have a mapping $\wp : \mathcal{V} \to \mathcal{W}$ such that

$\wp(\mathfrak{v}) = (\eta, \upsilon, \varsigma) - \lim_{n\to\infty} \wp_n(\mathfrak{v}) \in \mathfrak{L}$. Clearly $\wp(0) = \lim_{n\to\infty} \wp_n(0) = 0$.

Therefore $(\mathfrak{L}, \mathfrak{d})$ is a complete generalized metric space.

Theorem 3.2. Let $\Psi : \mathcal{V} \times \mathcal{V} \to \mathfrak{Z}$ be a function such that

$$\begin{cases} \eta\left(\Psi\left(\frac{\mathfrak{v}}{2}, \frac{\mathfrak{w}}{2}\right), \tau\right) \geq \eta'\left(\frac{\theta}{2}\Psi(\mathfrak{v}, \mathfrak{w}), \tau\right) \\ \upsilon\left(\Psi\left(\frac{\mathfrak{v}}{2}, \frac{\mathfrak{w}}{2}\right), \tau\right) \leq \upsilon'\left(\frac{\theta}{2}\Psi(\mathfrak{v}, \mathfrak{w}), \tau\right) \\ \varsigma\left(\Psi\left(\frac{\mathfrak{v}}{2}, \frac{\mathfrak{w}}{2}\right), \tau\right) \leq \varsigma'\left(\frac{\theta}{2}\Psi(\mathfrak{v}, \mathfrak{w}), \tau\right). \end{cases} \tag{3.2.1}$$

For some real θ with $0 < \theta < 1$, for all $\mathfrak{v}, \mathfrak{w} \in \mathcal{V}$, $\tau > 0$. If $\xi : \mathcal{V} \to \mathcal{W}$ be a mapping satisfying

$$\begin{cases} \eta(\mathcal{D}_1 \Psi(\mathfrak{v}, \mathfrak{w}), \tau) \geq \eta'(\Psi(\mathfrak{v}, \mathfrak{w}), \tau) \\ \upsilon(\mathcal{D}_1 \Psi(\mathfrak{v}, \mathfrak{w}), \tau) \leq \upsilon'(\Psi(\mathfrak{v}, \mathfrak{w}), \tau) \\ \varsigma(\mathcal{D}_1 \Psi(\mathfrak{v}, \mathfrak{w}), \tau) \leq \varsigma'(\Psi(\mathfrak{v}, \mathfrak{w}), \tau) \end{cases} \tag{3.2.2}$$

for all $\mathfrak{v}, \mathfrak{w} \in \mathcal{V}$, $\tau > 0$, where $\mathcal{D}_1 \Psi(\mathfrak{v}, \mathfrak{w})$ is given by (I).

Then there exists a unique additive mapping $\mathfrak{Y} : \mathcal{V} \to \mathcal{W}$ defined by

$\mathfrak{Y}(\mathfrak{v}) = (\eta, \upsilon, \varsigma) - \lim_{n\to\infty} 2^n \xi\left(\frac{\mathfrak{v}}{2^n}\right)$ for all $\mathfrak{v} \in \mathcal{V}$, $\tau > 0$ satisfying

$$\begin{cases} \eta(\mathfrak{Y}(\mathfrak{v}) - \xi(\mathfrak{v}), \tau) \geq \eta'\left(\Psi(\mathfrak{v}, \mathfrak{v}), \frac{2(1-\theta)}{\theta}\tau\right), \\ \upsilon(\mathfrak{Y}(\mathfrak{v}) - \xi(\mathfrak{v}), \tau) \leq \upsilon'\left(\Psi(\mathfrak{v}, \mathfrak{v}), \frac{2(1-\theta)}{\theta}\tau\right), \\ \varsigma(\mathfrak{Y}(\mathfrak{v}) - \xi(\mathfrak{v}), \tau) \leq \varsigma'\left(\Psi(\mathfrak{v}, \mathfrak{v}), \frac{2(1-\theta)}{\theta}\tau\right). \end{cases} \tag{3.2.3}$$

Proof. Putting $\mathfrak{w} = \mathfrak{v}$ in (3.2.2) we get

$$\begin{cases} \eta(\xi(2\mathfrak{v}) - 2\xi(\mathfrak{v}), \tau) \geq \eta'(\Psi(\mathfrak{v}, \mathfrak{v}), \tau), \\ \upsilon(\xi(2\mathfrak{v}) - 2\xi(\mathfrak{v}), \tau) \leq \upsilon'(\Psi(\mathfrak{v}, \mathfrak{v}), \tau), \\ \varsigma(\xi(2\mathfrak{v}) - 2\xi(\mathfrak{v}), \tau) \leq \varsigma'(\Psi(\mathfrak{v}, \mathfrak{v}), \tau). \end{cases} \tag{3.2.4}$$

Now consider the set $\mathfrak{L} = \{\wp : \mathcal{V} \to \mathcal{W};\ \wp(0) = 0\}$ and introduce a complete generalized metric on $\mathfrak{L}$ as per Lemma (3.1). Define a mapping $\mathfrak{T} : \mathfrak{L} \to \mathfrak{L}$ by $\mathfrak{T}\wp(\mathfrak{v}) = 2\wp\left(\frac{\mathfrak{v}}{2}\right)$ for all $\wp \in \mathfrak{L}$ and $\mathfrak{v} \in \mathcal{V}$. We now prove that $\mathfrak{T}$ is strictly contracting mapping of $\mathfrak{L}$ with the Lipschitz constant θ. Let $\wp, \mathscr{g} \in \mathfrak{L}$ and $\varepsilon > 0$.

Then there exists $\kappa' \in \mathbb{R}^+$ satisfying $\begin{cases} \eta(\wp(\mathfrak{v}) - \mathscr{g}(\mathfrak{v}), \kappa'\tau) \geq \eta'(\mathfrak{v}, \tau), \\ \upsilon(\wp(\mathfrak{v}) - \mathscr{g}(\mathfrak{v}), \kappa'\tau) \leq \upsilon'(\mathfrak{v}, \tau), \\ \varsigma(\wp(\mathfrak{v}) - \mathscr{g}(\mathfrak{v}), \kappa'\tau) \leq \varsigma'(\mathfrak{v}, \tau), \end{cases}$ such that $\mathfrak{d}(\wp, \mathscr{g}) \leq \kappa' < d(\wp, \mathscr{g}) + \varepsilon$.

Then,

$$\inf\left\{\kappa \in \mathbb{R}^+ : \begin{cases} \eta(\wp(\mathfrak{v}) - \mathscr{g}(\mathfrak{v}), \kappa\tau) \geq \eta'(\Psi(\mathfrak{v},\mathfrak{v}),\tau) \\ \upsilon(\wp(\mathfrak{v}) - \mathscr{g}(\mathfrak{v}), \kappa\tau) \leq \upsilon'(\Psi(\mathfrak{v},\mathfrak{v}),\tau) \text{ for all } \mathfrak{v} \in \mathcal{V}, \tau > 0 \\ \varsigma(\wp(\mathfrak{v}) - \mathscr{g}(\mathfrak{v}), \kappa\tau) \leq \varsigma'(\Psi(\mathfrak{v},\mathfrak{v}),\tau) \end{cases}\right\} \leq \kappa' d(\wp, \mathscr{g}) + \varepsilon.$$

That is,

$$\inf\left\{\kappa \in \mathbb{R}^+ : \begin{cases} \eta\left(2\wp\left(\frac{\mathfrak{v}}{2}\right) - 2\mathscr{g}\left(\frac{\mathfrak{v}}{2}\right), 2\kappa\tau\right) \geq \eta'\left(\Psi\left(\frac{\mathfrak{v}}{2},\frac{\mathfrak{v}}{2}\right),\tau\right), \\ \upsilon\left(2\wp\left(\frac{\mathfrak{v}}{2}\right) - 2\mathscr{g}\left(\frac{\mathfrak{v}}{2}\right), 2\kappa\tau\right) \leq \upsilon'\left(\Psi\left(\frac{\mathfrak{v}}{2},\frac{\mathfrak{v}}{2}\right),\tau\right), \text{ for all } \mathfrak{v} \in \mathcal{V}, \tau > 0 \\ \varsigma\left(2\wp\left(\frac{\mathfrak{v}}{2}\right) - 2\mathscr{g}(\mathfrak{v}), 2\kappa\tau\right) \leq \varsigma'\left(\Psi\left(\frac{\mathfrak{v}}{2},\frac{\mathfrak{v}}{2}\right),\tau\right), \end{cases}\right\}$$

$\leq \kappa' < d(\wp, \mathscr{g}) + \varepsilon.$

$$\inf\left\{\kappa \in \mathbb{R}^+ : \begin{cases} \eta(\mathfrak{T}\wp(\mathfrak{v}) - \mathfrak{T}\mathscr{g}(\mathfrak{v}), 2\kappa\tau) \geq \eta'\left(\frac{\theta}{2}\Psi(\mathfrak{v},\mathfrak{v}),\tau\right), \\ \upsilon(\mathfrak{T}\wp(\mathfrak{v}) - \mathfrak{T}\mathscr{g}(\mathfrak{v}), 2\kappa\tau) \leq \upsilon'\left(\frac{\theta}{2}\Psi(\mathfrak{v},\mathfrak{v}),\tau\right), \text{for all } \mathfrak{v} \in \mathcal{V}, \tau > 0 \\ \varsigma(\mathfrak{T}\wp(\mathfrak{v}) - \mathfrak{T}\mathscr{g}(\mathfrak{v}), 2\kappa\tau) \leq \varsigma'\left(\frac{\theta}{2}\Psi(\mathfrak{v},\mathfrak{v}),\tau\right), \end{cases}\right\}$$

$< d(\wp, \mathscr{g}) + \varepsilon.$

$$\inf\left\{\kappa \in \mathbb{R}^+ : \begin{cases} \eta(\mathfrak{T}\wp(\mathfrak{v}) - \mathfrak{T}\mathscr{g}(\mathfrak{v}), \theta\kappa\tau) \geq \eta'(\Psi(\mathfrak{v},\mathfrak{v}),\tau), \\ \upsilon(\mathfrak{T}\wp(\mathfrak{v}) - \mathfrak{T}\mathscr{g}(\mathfrak{v}), \theta\kappa\tau) \leq \upsilon'(\Psi(\mathfrak{v},\mathfrak{v}),\tau), \text{ for all } \mathfrak{v} \in \mathcal{V}, \tau > 0 \\ \varsigma(\mathfrak{T}\wp(\mathfrak{v}) - \mathfrak{T}\mathscr{g}(\mathfrak{v}), \theta\kappa\tau) \leq \varsigma'(\Psi(\mathfrak{v},\mathfrak{v}),\tau), \end{cases}\right\}$$

$< d(\wp, \mathscr{g}) + \varepsilon$ or, $\mathfrak{d}\{\frac{1}{\theta}(\mathfrak{T}\wp, \mathfrak{T}\mathscr{g})\} < d(\wp, \mathscr{g}) + \varepsilon$ or, $\mathfrak{d}\{(\mathfrak{T}\wp, \mathfrak{T}\mathscr{g})\} < \theta\{\mathfrak{d}(\wp, \mathscr{g}) + \varepsilon\}$

Taking $\varepsilon \to 0$ we get $\mathfrak{d}(\mathfrak{T}\wp, \mathfrak{T}\mathscr{g}) \leq \theta\, \mathfrak{d}(\wp, \mathscr{g})$. Therefore $\mathfrak{T}$ is strictly contractive mapping with Lipschitz constant $\theta < 1$. Also form (3.2.4),

$$\begin{cases} \eta\left(\xi(\mathfrak{v}) - 2\xi\left(\frac{\mathfrak{v}}{2}\right), \frac{\theta}{2}\tau\right) \geq \eta'(\Psi(\mathfrak{v},\mathfrak{v}),\tau), \\ \upsilon\left(\xi(\mathfrak{v}) - 2\xi\left(\frac{\mathfrak{v}}{2}\right), \frac{\theta}{2}\tau\right) \leq \upsilon'(\Psi(\mathfrak{v},\mathfrak{v}),\tau), \\ \varsigma\left(\xi(\mathfrak{v}) - 2\xi\left(\frac{\mathfrak{v}}{2}\right), \frac{\theta}{2}\tau\right) \leq \varsigma'(\Psi(\mathfrak{v},\mathfrak{v}),\tau). \end{cases}$$

Therefore $\mathfrak{d}(\xi, \mathfrak{T}\xi) \leq \frac{\theta}{2}$. Again replacing $\mathfrak{v}$ by $2^{-(n+1)}\mathfrak{v}$ in (3.2.4) we get

$$\begin{cases}\eta\left(2^n\xi\left(\frac{\mathfrak{v}}{2^n}\right)-2\xi\left(\frac{\mathfrak{v}}{2^{n+1}}\right),2^n\tau\right)\geq\eta'\left(\Psi\left(\frac{\mathfrak{v}}{2^{n+1}},\frac{\mathfrak{v}}{2^{n+1}}\right),\tau\right)\geq\eta'\left(\left(\frac{\theta}{2}\right)^{n+1}\Psi(\mathfrak{v},\mathfrak{v}),\tau\right),\\ \upsilon\left(2^n\xi\left(\frac{\mathfrak{v}}{2^n}\right)-2\xi\left(\frac{\mathfrak{v}}{2^{n+1}}\right),2^n\tau\right)\leq\upsilon'\left(\Psi\left(\frac{\mathfrak{v}}{2^{n+1}},\frac{\mathfrak{v}}{2^{n+1}}\right),\tau\right)\leq\upsilon'\left(\left(\frac{\theta}{2}\right)^{n+1}\Psi(\mathfrak{v},\mathfrak{v}),\tau\right),\\ \varsigma\left(2^n\xi\left(\frac{\mathfrak{v}}{2^n}\right)-2\xi\left(\frac{\mathfrak{v}}{2^{n+1}}\right),2^n\tau\right)\leq\varsigma'\left(\Psi\left(\frac{\mathfrak{v}}{2^{n+1}},\frac{\mathfrak{v}}{2^{n+1}}\right),\tau\right)\leq\varsigma'\left(\left(\frac{\theta}{2}\right)^{n+1}\Psi(\mathfrak{v},\mathfrak{v}),\tau\right).\end{cases}$$

Or $\begin{cases}\eta\left(\mathfrak{T}^n\xi(\mathfrak{v})-\mathfrak{T}^{n+1}\xi(\mathfrak{v}),\frac{\theta^{n+1}}{2}\tau\right)\geq\eta'(\Psi(\mathfrak{v},\mathfrak{v}),\tau),\\ \upsilon\left(\mathfrak{T}^n\xi(\mathfrak{v})-\mathfrak{T}^{n+1}\xi(\mathfrak{v}),\frac{\theta^{n+1}}{2}\tau\right)\leq\upsilon'(\Psi(\mathfrak{v},\mathfrak{v}),\tau),\\ \varsigma\left(\mathfrak{T}^n\xi(\mathfrak{v})-\mathfrak{T}^{n+1}\xi(\mathfrak{v}),\frac{\theta^{n+1}}{2}\tau\right)\leq\varsigma'(\Psi(\mathfrak{v},\mathfrak{v}),\tau).\end{cases}$

Hence $\mathfrak{d}(\mathfrak{T}^{n+1}\xi,\mathfrak{T}^n\xi)\leq\frac{\theta^{n+1}}{2}<\infty$, Lipschitz constant $\theta<1$ for $n\geq m_0=1$.

Therefore by Theorem (2.9) there exists a mapping $\mathfrak{Y}:\mathcal{V}\to\mathcal{W}$ fulfilling the following:

1. $\mathfrak{Y}$ is a fixed point of $\mathfrak{T}$, that is, $\mathfrak{Y}(\mathfrak{v})=2\mathfrak{Y}\left(\frac{\mathfrak{v}}{2}\right)$ for all $\mathfrak{v}\in\mathcal{V}$. The mapping $\mathfrak{Y}$ is a unique fixed point of $\mathfrak{T}$ in the set $\mathfrak{L}_1=\{\wp\in\mathfrak{L}:\ \mathfrak{d}(\mathfrak{T}^{n_0}\xi,\wp)=\mathfrak{d}(\mathfrak{T}\xi,\wp)<\infty$.

Therefore $\mathfrak{d}(\mathfrak{T}\xi,\mathfrak{Y})<\infty$. Also from (3.4), $\mathfrak{d}(\mathfrak{T}\xi,\xi)\leq\frac{\theta}{2}<\infty$. Thus $\xi\in\mathfrak{L}_1$.

Now, $\mathfrak{d}(\xi,\mathfrak{Y})\leq\max\{\mathfrak{d}(\xi,\mathfrak{T}\xi),\mathfrak{d}(\mathfrak{T}\xi,)\}<\infty$. Thus, there exists $\kappa\in(0,\infty)$ satisfying

$$\begin{cases}\eta(\xi(\mathfrak{v})-\mathfrak{Y}(\mathfrak{v}),\kappa\tau)\geq\eta'(\Psi(\mathfrak{v},\mathfrak{v}),\tau)\\ \upsilon(\xi(\mathfrak{v})-\mathfrak{Y}\xi(\mathfrak{v}),\kappa\tau)\leq\upsilon'(\Psi(\mathfrak{v},\mathfrak{v}),\tau)\\ \varsigma(\xi(\mathfrak{v})-\mathfrak{Y}\xi(\mathfrak{v}),\kappa\tau)\leq\varsigma'(\Psi(\mathfrak{v},\mathfrak{v}),\tau)\end{cases}\quad\text{for all } \mathfrak{v}\in\mathcal{V},\tau>0 \tag{3.2.5}$$

Also, from (3.2.5) we have,

$$\begin{cases}\eta(\xi\left(\frac{\mathfrak{v}}{2^n}\right)-\mathfrak{Y}\left(\frac{\mathfrak{v}}{2^n}\right),\kappa\tau)\geq\eta'(\Psi\left(\frac{\mathfrak{v}}{2^n},\frac{\mathfrak{v}}{2^n}\right),\tau,\\ \upsilon\left(\xi\left(\frac{\mathfrak{v}}{2^n}\right)-\mathfrak{Y}\left(\frac{\mathfrak{v}}{2^n}\right),\kappa\tau\right)\leq\upsilon'\left(\Psi\left(\frac{\mathfrak{v}}{2^n},\frac{\mathfrak{v}}{2^n}\right),\tau\right),\\ \varsigma\left(\xi\left(\frac{\mathfrak{v}}{2^n}\right)-\mathfrak{Y}\left(\frac{\mathfrak{v}}{2^n}\right),\kappa\tau\right)\leq\varsigma'\left(\Psi\left(\frac{\mathfrak{v}}{2^n},\frac{\mathfrak{v}}{2^n}\right),\tau\right).\end{cases}$$

Or $\begin{cases}\eta\left(2^n\xi\left(\frac{\mathfrak{v}}{2^n}\right)-2^n\mathfrak{Y}\left(\frac{\mathfrak{v}}{2^n}\right),2^n\kappa\tau\right)\geq\eta'\left(\Psi\left(\frac{\mathfrak{v}}{2^n},\frac{\mathfrak{v}}{2^n}\right),\tau\right)\geq\eta'\left(\frac{\theta^n}{2^n}\Psi(\mathfrak{v},\mathfrak{v}),\tau\right),\\ \upsilon\left(2^n\xi\left(\frac{\mathfrak{v}}{2^n}\right)-2^n\mathfrak{Y}\left(\frac{\mathfrak{v}}{2^n}\right),2^n\kappa\tau\right)\leq\upsilon'\left(\Psi\left(\frac{\mathfrak{v}}{2^n},\frac{\mathfrak{v}}{2^n}\right),\tau\right)\leq\upsilon'\left(\frac{\theta^n}{2^n}\Psi(\mathfrak{v},\mathfrak{v}),\tau\right),\\ \varsigma\left(2^n\xi\left(\frac{\mathfrak{v}}{2^n}\right)-2^n\mathfrak{Y}\left(\frac{\mathfrak{v}}{2^n}\right),2^n\kappa\tau\right)\leq\varsigma'\left(\Psi\left(\frac{\mathfrak{v}}{2^n},\frac{\mathfrak{v}}{2^n}\right),\tau\right)\leq\varsigma'\left(\frac{\theta^n}{2^n}\Psi(\mathfrak{v},\mathfrak{v}),\tau\right).\end{cases}$

That is, $\begin{cases} \eta(\mathfrak{T}^n\xi(\mathfrak{v}) - \mathfrak{Y}(\mathfrak{v}), \theta^n\kappa\tau) \geq \eta'(\Psi(\mathfrak{v},\mathfrak{v}),\tau), \\ \upsilon(\mathfrak{T}^n\xi(\mathfrak{v}) - \mathfrak{Y}(\mathfrak{v}), \theta^n\kappa\tau) \leq \upsilon'(\Psi(\mathfrak{v},\mathfrak{v}),\tau), \\ \varsigma(\mathfrak{T}^n\xi(\mathfrak{v}) - \mathfrak{Y}\xi(\mathfrak{v}), \theta^n\kappa\tau) \leq \varsigma'(\Psi(\mathfrak{v},\mathfrak{v}),\tau). \end{cases}$

As $\mathfrak{Y}(\mathfrak{v}) = 2\mathfrak{Y}\left(\frac{\mathfrak{v}}{2}\right) = 2^2\mathfrak{Y}\left(\frac{\mathfrak{v}}{2^2}\right) = \cdots = 2^n\mathfrak{Y}\left(\frac{\mathfrak{v}}{2^n}\right)$.

2. $\mathfrak{d}(\mathfrak{T}^n\xi, \mathfrak{Y}) =$

$$\inf\left\{\kappa \in \mathbb{R}^+ : \begin{cases} \eta(\mathfrak{T}^n\xi(\mathfrak{v}) - \mathfrak{Y}(\mathfrak{v}), \theta^n\kappa\tau) \geq \eta'(\Psi(\mathfrak{v},\mathfrak{v}),\tau) \\ \upsilon(\mathfrak{T}^n\xi(\mathfrak{v}) - \mathfrak{Y}(\mathfrak{v}), \theta^n\kappa\tau) \leq \upsilon'(\Psi(\mathfrak{v},\mathfrak{v}),\tau) \text{ for all } \mathfrak{v} \in \mathcal{V}, \tau > 0 \\ \varsigma(\mathfrak{T}^n\xi(\mathfrak{v}) - \mathfrak{Y}(\mathfrak{v}), \theta^n\kappa\tau) \leq \varsigma'(\Psi(\mathfrak{v},\mathfrak{v}),\tau) \end{cases}\right\}.$$

Therefore $\mathfrak{d}(\mathfrak{T}^n\xi, \mathfrak{Y}) \leq \theta^n\kappa \to 0$ as $n \to \infty$ and $\theta < 1$. This implies the inequality

$$\mathfrak{Y}(\mathfrak{v}) = (\eta,\upsilon,\varsigma) - \lim_{n\to\infty} \mathfrak{T}^n\xi(\mathfrak{v}) = (\eta,\upsilon,\varsigma) - \lim_{n\to\infty} 2^n\xi\left(\frac{\mathfrak{v}}{2^n}\right) \text{ for all } \mathfrak{v} \in \mathcal{V}. \qquad (3.2.6)$$

3. $\mathfrak{d}(\xi, \mathfrak{Y}) \leq \frac{1}{1-\mathfrak{J}}\mathfrak{d}(\xi, \mathfrak{T}\xi)$ with $\xi \in \mathfrak{L}_1$ which implies the inequality

$\mathfrak{d}(\xi, \mathfrak{Y}) \leq \frac{1}{1-\theta} \times \frac{\theta}{2} = \frac{\theta}{2(1-\theta)}$. It follows the results (3.2.3).

Now, replacing $\mathfrak{v}$ and $\mathfrak{w}$ by $2^{-n}\mathfrak{v}$ and $2^{-n}\mathfrak{w}$ in (3.2.2) we have

$$\begin{cases} \eta(2^n\xi\left(\frac{\mathfrak{v}+\mathfrak{w}}{2^n}\right) - 2^n\xi\left(\frac{\mathfrak{v}}{2^n}\right) - 2^n\xi\left(\frac{\mathfrak{w}}{2^n}\right) - \rho(2^n\xi\left(\frac{\mathfrak{v}+\mathfrak{w}}{2^{n+1}}\right) - 2^n\xi\left(\frac{\mathfrak{v}}{2^n}\right) - 2^n\xi\left(\frac{\mathfrak{w}}{2^n}\right)),\tau) \geq \eta'(\Psi(\mathfrak{v},\mathfrak{v}), \frac{\tau}{\theta^n}), \\ \upsilon(2^n\xi\left(\frac{\mathfrak{v}+\mathfrak{w}}{2^n}\right) - 2^n\xi\left(\frac{\mathfrak{v}}{2^n}\right) - 2^n\xi\left(\frac{\mathfrak{w}}{2^n}\right) - \rho(2^n\xi\left(\frac{\mathfrak{v}+\mathfrak{w}}{2^{n+1}}\right) - 2^n\xi\left(\frac{\mathfrak{v}}{2^n}\right) - 2^n\xi\left(\frac{\mathfrak{w}}{2^n}\right)),\tau) \leq \upsilon'(\Psi(\mathfrak{v},\mathfrak{v}), \frac{\tau}{\theta^n}), \\ \varsigma(2^n\xi\left(\frac{\mathfrak{v}+\mathfrak{w}}{2^n}\right) - 2^n\xi\left(\frac{\mathfrak{v}}{2^n}\right) - 2^n\xi\left(\frac{\mathfrak{w}}{2^n}\right) - \rho(2^n\xi\left(\frac{\mathfrak{v}+\mathfrak{w}}{2^{n+1}}\right) - 2^n\xi\left(\frac{\mathfrak{v}}{2^n}\right) - 2^n\xi\left(\frac{\mathfrak{w}}{2^n}\right)),\tau) \leq \varsigma'(\Psi(\mathfrak{v},\mathfrak{v}), \frac{\tau}{\theta^n}). \end{cases}$$

(3.2.7

Taking limit as $n \to \infty$ in (3.7) and using $\theta < 1$, $\eta(\mathfrak{v},\tau) = 1 \Leftrightarrow \mathfrak{v} = 0, \tau > 0, \upsilon(\mathfrak{v},\tau) = 1 \Leftrightarrow \mathfrak{v} = 0, \tau > 0, \varsigma(\mathfrak{v},\tau) = 1 \Leftrightarrow \mathfrak{v} = 0, \tau > 0$, (2.2) we obtain,

$$\mathfrak{Y}(\mathfrak{v}+\mathfrak{w}) - \mathfrak{Y}(\mathfrak{v}) - \mathfrak{Y}(\mathfrak{w}) = \rho(2\mathfrak{Y}\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right) - \mathfrak{Y}(\mathfrak{v}) - \mathfrak{Y}(\mathfrak{w})) \qquad (3.2.8)$$

Therefore by the Lemma (2.10), we can say that $\mathfrak{Y} : \mathcal{V} \to \mathcal{W}$ is additive. The uniqueness of $\mathfrak{Y}$ follows from the fact that $\mathfrak{Y}$ is the unique fixed point of $\mathfrak{T}$ by the Theorem (2.9). This completes the proof of the theorem.

Corollary: 3.3. Let $\mathcal{V}$ be non-Archimedean normed space with norm $\|.\|$ over the non-Archimedean field $\mathcal{F}$ and $q < 1$ be a non-negative real number, $\mathfrak{z}_0 \in \mathfrak{Z}$ and let $\xi : \mathcal{V} \to \mathcal{W}$ be a mapping such that

$$\begin{cases} \eta(\mathcal{D}_1\xi(\mathfrak{v},\mathfrak{w}),\tau) \geq \eta'(\mathfrak{z}_0(\|\mathfrak{v}\|^q + \|\mathfrak{w}\|^q),\tau), \\ \upsilon(\mathcal{D}_1\xi(\mathfrak{v},\mathfrak{w}),\tau) \leq \upsilon'(\mathfrak{z}_0(\|\mathfrak{v}\|^q + \|\mathfrak{w}\|^q),\tau), \\ \varsigma(\mathcal{D}_1\xi(\mathfrak{v},\mathfrak{w}),\tau) \leq \varsigma'(\mathfrak{z}_0(\|\mathfrak{v}\|^q + \|\mathfrak{w}\|^q),\tau). \end{cases} \tag{3.3.1}$$

($\mathfrak{v} \in \mathcal{V}, \tau > 0$), where $\mathcal{D}_1\xi(\mathfrak{v},\mathfrak{w})$ is given by (I). Then there exists a unique additive mapping $\mathfrak{Y} : \mathcal{V} \to \mathcal{W}$ for all $\mathfrak{v} \in \mathcal{V}, \tau > 0$ satisfying

$$\begin{cases} \eta(\mathfrak{Y}(\mathfrak{v}) - \xi(\mathfrak{v}),\tau) \geq \eta'\left(\mathfrak{z}_0\|\mathfrak{v}\|^q, \frac{(2)^q-2}{2}\tau\right), \\ \upsilon(\mathfrak{Y}(\mathfrak{v}) - \xi(\mathfrak{v}),\tau) \leq \upsilon'\left(\mathfrak{z}_0\|\mathfrak{v}\|^q, \frac{(2)^q-2}{2}\tau\right), \\ \varsigma(\mathfrak{Y}(\mathfrak{v}) - \xi(\mathfrak{v}),\tau) \leq \varsigma'\left(\mathfrak{z}_0\|\mathfrak{v}\|^q, \frac{(2)^q-2}{2}\tau\right). \end{cases} \tag{3.3.2}$$

Proof. Define $\Psi(\mathfrak{v},\mathfrak{w}) = \mathfrak{z}_0(\|\mathfrak{v}\|^q + \|\mathfrak{w}\|^q)$ and the proof derived from Theorem (3.2) by taking $\theta = 2^{q-1}$.

Theorem 3.4. Let $\Psi : \mathcal{V} \times \mathcal{V} \to \mathfrak{Z}$ be a function such that

$$\begin{cases} \eta(\Psi(\mathfrak{v},\mathfrak{w}),\tau) \geq \eta'\left(2\theta\Psi\left(\frac{\mathfrak{v}}{2},\frac{\mathfrak{w}}{2}\right),\tau\right) \\ \upsilon(\Psi(\mathfrak{v},\mathfrak{w}),\tau) \leq \upsilon'\left(2\theta\Psi\left(\frac{\mathfrak{v}}{2},\frac{\mathfrak{w}}{2}\right),\tau\right) \\ \varsigma(\Psi(\mathfrak{v},\mathfrak{w}),\tau) \leq \varsigma'\left(2\theta\Psi\left(\frac{\mathfrak{v}}{2},\frac{\mathfrak{w}}{2}\right),\tau\right) \end{cases} \tag{3.4.1}$$

for some real θ with $0 < \theta < 1$, $\mathfrak{v},\mathfrak{w} \in \mathcal{V}$, $\tau > 0$. If $\xi : \mathcal{V} \to \mathcal{W}$ be a mapping satisfying (3.2.2) then there exists a unique additive mapping $\mathfrak{Y} : \mathcal{V} \to \mathcal{W}$ defined by $\mathfrak{Y}(\mathfrak{v}) = (\eta,\upsilon,\varsigma) - \lim_{n\to\infty} \frac{1}{2^n}\xi(2^n\mathfrak{v})$ for all $\mathfrak{v} \in \mathcal{V}, \tau > 0$ satisfying

$$\begin{cases} \eta(\mathfrak{Y}(\mathfrak{v}) - \xi(\mathfrak{v}),\tau) \geq \eta'(\Psi(\mathfrak{v},\mathfrak{v}), 2(1-\theta),\tau), \\ \upsilon(\mathfrak{Y}(\mathfrak{v}) - \xi(\mathfrak{v}),\tau) \leq \upsilon'(\Psi(\mathfrak{v},\mathfrak{v}), 2(1-\theta),\tau), \\ \varsigma(\mathfrak{Y}(\mathfrak{v}) - \xi(\mathfrak{v}),\tau) \leq \varsigma'(\Psi(\mathfrak{v},\mathfrak{v}), 2(1-\theta),\tau). \end{cases} \tag{3.4.2}$$

Proof. Putting $\mathfrak{w} = \mathfrak{v}$ in (3.2.2) we get

$$\begin{cases} \eta\left(\xi(\mathfrak{v}) - \frac{1}{2}\xi(2\mathfrak{v}),\frac{\tau}{2}\right) \geq \eta'(\Psi(\mathfrak{v},\mathfrak{v}),\tau), \\ \upsilon\left(\xi(\mathfrak{v}) - \frac{1}{2}\xi(2\mathfrak{v}),\frac{\tau}{2}\right) \leq \upsilon'(\Psi(\mathfrak{v},\mathfrak{v}),\tau), \\ \varsigma\left(\xi(\mathfrak{v}) - \frac{1}{2}\xi(2\mathfrak{v}),\frac{\tau}{2}\right) \leq \varsigma'(\Psi(\mathfrak{v},\mathfrak{v}),\tau). \end{cases} \tag{3.4.3}$$

As before consider the set $\mathfrak{L} = \{\wp : \mathcal{V} \to \mathcal{W};\ \wp(0) = 0\}$ and introduce a complete generalized metric on $\mathfrak{L}$ such that $\mathfrak{T} : \mathfrak{L} \to \mathfrak{L}$ by $\mathfrak{T}\wp(\mathfrak{v}) = \frac{1}{2}\wp(2\mathfrak{v})$ for all $\wp \in \mathfrak{L}$ and $\mathfrak{v} \in \mathcal{V}$.

Also, from (3.4.3) we see that $\mathfrak{d}(\xi, \mathfrak{T}\xi) \le \frac{1}{2}$. As before we conclude that the result (3.4.2).

Corollary: 3.5. Let $q > 1$ be a non-negative real number, $\mathcal{V}$ be a *NANS* with norm $\|.\|$ over the non-Archimedean field $\mathcal{F}$. Let $\mathfrak{z}_0 \in \mathfrak{Z}$ and let $\xi : \mathcal{V} \to \mathcal{W}$ be a mapping satisfying (3.3.1). Then there exists unique additive mapping $\mathfrak{Y} : \mathcal{V} \to \mathcal{W}$ for all $\mathfrak{v} \in \mathcal{V}, \tau > 0$ satisfying

$$\begin{cases} \eta(\mathfrak{Y}(\mathfrak{v}) - \xi(\mathfrak{v}), \tau) \ge \eta'\left(\mathfrak{z}_0\|\mathfrak{v}\|^q, \frac{(2)^q - 2}{2}\tau\right), \\ \upsilon(\mathfrak{Y}(\mathfrak{v}) - \xi(\mathfrak{v}), \tau) \le \upsilon'\left(\mathfrak{z}_0\|\mathfrak{v}\|^q, \frac{(2)^q - 2}{2}\tau\right), \\ \varsigma(\mathfrak{Y}(\mathfrak{v}) - \xi(\mathfrak{v}), \tau) \le \varsigma'\left(\mathfrak{z}_0\|\mathfrak{v}\|^q, \frac{(2)^q - 2}{2}\tau\right). \end{cases} \tag{3.5.1}$$

Proof. Define $\Psi(\mathfrak{v}, \mathfrak{w}) = \mathfrak{z}_0(\|\mathfrak{v}\|^q + \|\mathfrak{w}\|^q)$ and the proof derived from Theorem (3.4) by taking $\theta = 2^{q-1}$.

4. Non-Archimedean Neutrosophic Stability Of Additive ρ – Functional Equation (II)

Theorem 4.1. Let $\Psi : \mathcal{V} \times \mathcal{V} \to \mathfrak{Z}$ be a function satisfying (3.2.1). If $\xi : \mathcal{V} \to \mathcal{W}$ be a mapping with $\xi(0) = 0$ satisfying

$$\begin{cases} \eta(\mathcal{D}_2\xi(\mathfrak{v}, \mathfrak{w}), \tau) \ge \eta'(\Psi(\mathfrak{v}, \mathfrak{w}), \tau), \\ \upsilon(\mathcal{D}_2\xi(\mathfrak{v}, \mathfrak{w}), \tau) \le \upsilon'(\Psi(\mathfrak{v}, \mathfrak{w}), \tau), \\ \varsigma(\mathcal{D}_2\xi(\mathfrak{v}, \mathfrak{w}), \tau) \le \varsigma'(\Psi(\mathfrak{v}, \mathfrak{w}), \tau). \end{cases} \tag{4.1.1}$$

$(\mathfrak{v} \in \mathcal{V}, \tau > 0)$, where $\mathcal{D}_2$ is given by (II). Then there exists a unique additive mapping $\mathfrak{Y} : \mathcal{V} \to \mathcal{W}$ defined by $\mathfrak{Y}(\mathfrak{v}) = (\eta, \upsilon, \varsigma) - \lim_{n\to\infty} \frac{1}{2^n}\xi(\frac{\mathfrak{v}}{2^n})$ for all $\mathfrak{v} \in \mathcal{V}, \tau > 0$ satisfying

$$\begin{cases} \eta(\mathfrak{Y}(\mathfrak{v}) - \xi(\mathfrak{v}), \tau) \ge \eta'(\Psi(\mathfrak{v}, 0), (1-\theta)\tau) \\ \upsilon(\mathfrak{Y}(\mathfrak{v}) - \xi(\mathfrak{v}), \tau) \le \upsilon'(\Psi(\mathfrak{v}, 0), (1-\theta)\tau) \\ \varsigma(\mathfrak{Y}(\mathfrak{v}) - \xi(\mathfrak{v}), \tau) \le \varsigma'(\Psi(\mathfrak{v}, 0), (1-\theta)\tau) \end{cases} \tag{4.1.2}$$

Proof. Putting $\mathfrak{w} = 0$ in (4.1.1) and the rest of the proof follows from the Theorem (3.2).

Example 4.2. Let $: \mathcal{V} = \mathcal{W} = \mathfrak{Z} = \mathbb{R}$. Define $\xi : \mathbb{R} \to \mathbb{R}$ by $\xi(\mathfrak{v}) = \mathfrak{v}$. Let $\eta(\mathfrak{v}, \tau) = \eta'(\mathfrak{v}, \tau) = \frac{\tau}{\tau + |\mathfrak{v}|}$, $\upsilon(\mathfrak{v}, \tau) = \upsilon'(\mathfrak{v}, \tau) = \frac{|\mathfrak{v}|}{\tau + |\mathfrak{v}|}$ and $\varsigma(\mathfrak{v}, \tau) = \varsigma'(\mathfrak{v}, \tau) = \frac{|\mathfrak{v}|}{\tau}$, for all $\mathfrak{v} \in \mathcal{V}, \tau > 0$. Let $\Psi(\mathfrak{v}, \mathfrak{w}) = \|\mathfrak{v}\|^2 + \|\mathfrak{w}\|^2$. Clearly, the functions Ψ and ξ satisfies the conditions (3.2.1) and (4.1.1). Then, the unique additive function $\mathfrak{Y}(\mathfrak{v}) = (\eta, \upsilon, \varsigma) - \lim_{n\to\infty} \frac{1}{2^n} \xi(\frac{\mathfrak{v}}{2^n})$, for all $\mathfrak{v} \in \mathcal{V}, \tau > 0$, exists that has satisfied the condition (4.1.2).

Corollary 4.3. Let $q < 1$ be a non-negative real number and $\mathcal{V}$ be a *NANS* with norm $\|.\|$ over the non-Archimedean field $\mathcal{F}$. Let $\mathfrak{z}_0 \in \mathfrak{Z}$ and let $\xi : \mathcal{V} \to \mathcal{W}$ be a mapping with $\xi(0) = 0$ such that

$$\begin{cases} \eta(\mathcal{D}_2\xi(\mathfrak{v}, \mathfrak{w}), \tau) \geq \eta'(\mathfrak{z}_0(\|\mathfrak{v}\|^q + \|\mathfrak{w}\|^q), \tau) \\ \upsilon(\mathcal{D}_2\xi(\mathfrak{v}, \mathfrak{w}), \tau) \leq \upsilon'(\mathfrak{z}_0(\|\mathfrak{v}\|^q + \|\mathfrak{w}\|^q), \tau) \\ \varsigma(\mathcal{D}_2\xi(\mathfrak{v}, \mathfrak{w}), \tau) \leq \varsigma'(\mathfrak{z}_0(\|\mathfrak{v}\|^q + \|\mathfrak{w}\|^q), \tau) \end{cases} \tag{4.3.1}$$

$(\mathfrak{v} \in \mathcal{V}, \tau > 0)$, where $\mathcal{D}_2$ is given by (II). Then there exists a unique additive mapping $\mathfrak{Y} : \mathcal{V} \to \mathcal{W}$ for all $\mathfrak{v} \in \mathcal{V}, \tau > 0$ satisfying

$$\begin{cases} \eta(\mathfrak{Y}(\mathfrak{v}) - \xi(\mathfrak{v}), \tau) \geq \eta'\left(\mathfrak{z}_0\|\mathfrak{v}\|^q, \frac{(2)^q - 2}{2}\tau\right), \\ \upsilon(\mathfrak{Y}(\mathfrak{v}) - \xi(\mathfrak{v}), \tau) \leq \upsilon'\left(\mathfrak{z}_0\|\mathfrak{v}\|^q, \frac{(2)^q - 2}{2}\tau\right), \\ \varsigma(\mathfrak{Y}(\mathfrak{v}) - \xi(\mathfrak{v}), \tau) \leq \varsigma'\left(\mathfrak{z}_0\|\mathfrak{v}\|^q, \frac{(2)^q - 2}{2}\tau\right). \end{cases} \tag{4.3.2}$$

Proof. Define $\Psi(\mathfrak{v}, \mathfrak{w}) = \mathfrak{z}_0(\|\mathfrak{v}\|^q + \|\mathfrak{w}\|^q)$ and the proof derived from Theorem (4.1) by taking $\theta = 2^{1-q}$.

Theorem 4.4. Let $\Psi : \mathcal{V} \times \mathcal{V} \to \mathfrak{Z}$ be a function satisfying (3.4.1). If $\xi : \mathcal{V} \to \mathcal{W}$ be a mapping with $\xi(0) = 0$ satisfying (4.1.1), where $\mathcal{D}_2$ is given by (II). Then there exists a unique additive mapping $\mathfrak{Y} : \mathcal{V} \to \mathcal{W}$ for all $\mathfrak{v} \in \mathcal{V}, \tau > 0$ satisfying

$$\begin{cases} \eta(\mathfrak{Y}(\mathfrak{v}) - \xi(\mathfrak{v}), \tau) \geq \eta'\left(\Psi(\mathfrak{v}, 0), \frac{(1-\theta)}{\theta}\tau\right), \\ \upsilon(\mathfrak{Y}(\mathfrak{v}) - \xi(\mathfrak{v}), \tau) \leq \upsilon'\left(\Psi(\mathfrak{v}, 0), \frac{(1-\theta)}{\theta}\tau\right), \\ \varsigma(\mathfrak{Y}(\mathfrak{v}) - \xi(\mathfrak{v}), \tau) \leq \varsigma'\left(\Psi(\mathfrak{v}, 0), \frac{(1-\theta)}{\theta}\tau\right). \end{cases} \tag{4.4.1}$$

Proof. Putting $\mathfrak{w} = 0$ in (4.1) we get $\begin{cases} \eta(\xi(\mathfrak{v}) - \frac{1}{2}\xi(2\mathfrak{v}), \theta\tau) \geq \eta'(\Psi(\mathfrak{v}, 0), \tau) \\ \upsilon(\xi(\mathfrak{v}) - \frac{1}{2}\xi(2\mathfrak{v}), \theta\tau) \leq \upsilon'(\Psi(\mathfrak{v}, 0), \tau) \\ \varsigma(\xi(\mathfrak{v}) - \frac{1}{2}\xi(2\mathfrak{v}), \theta\tau) \leq \varsigma'(\Psi(\mathfrak{v}, 0), \tau) \end{cases}$

(4.4.2)

Hence $\mathfrak{d}(\xi, \mathfrak{T}\xi) \leq \theta$. Similarly as before we conclude the result (4.4.1).

Corollary 4.5. Let $q > 1$ be a non-negative real number and $\mathcal{V}$ be a *NANS* with norm $\|.\|$ over the non-Archimedean field $\mathcal{F}$, $\mathfrak{z}_0 \in \mathfrak{Z}$ and let $\xi : \mathcal{V} \to \mathcal{W}$ be a mapping with $\xi(0) = 0$ such that (4.3.1) hold. Then there exists a unique additive mapping $\mathfrak{Y} : \mathcal{V} \to \mathcal{W}$ for all $\mathfrak{v} \in \mathcal{V}, \tau > 0$ satisfying

$$\begin{cases} \eta(\mathfrak{Y}(\mathfrak{v}) - \xi(\mathfrak{v}), \tau) \geq \eta'\left(\mathfrak{z}_0 \|\mathfrak{v}\|^q, \frac{(2)^q - 2}{2}\tau\right), \\ \upsilon(\mathfrak{Y}(\mathfrak{v}) - \xi(\mathfrak{v}), \tau) \leq \upsilon'\left(\mathfrak{z}_0 \|\mathfrak{v}\|^q, \frac{(2)^q - 2}{2}\tau\right), \\ \varsigma(\mathfrak{Y}(\mathfrak{v}) - \xi(\mathfrak{v}), \tau) \leq \varsigma'\left(\mathfrak{z}_0 \|\mathfrak{v}\|^q, \frac{(2)^q - 2}{2}\tau\right). \end{cases} \tag{4.5.1}$$

Proof: Define $\Psi(\mathfrak{v}, \mathfrak{w}) = \mathfrak{z}_0(\|\mathfrak{v}\|^q + \|\mathfrak{w}\|^q)$ and the proof derived from Theorem (4.4) by taking $\theta = 2^{q-1}$.

5. Hyers -Ulam- Rassias Stability of the Additive ρ −Functional Equations in Neutrosophic Banach Spaces

Theorem 5.1. Let $\mathcal{V}$ be linear space, $(\mathfrak{Z}, \eta', \upsilon', \varsigma')$ be a neutrosophic normed space [*NNS*],

$\Psi : \mathcal{V} \times \mathcal{V} \to \mathfrak{Z}$ be a function such that

$$\begin{cases} \eta\left(\Psi\left(\frac{\mathfrak{v}}{2}, \frac{\mathfrak{w}}{2}\right), \tau\right) \geq \eta'\left(\frac{\theta}{2}\Psi(\mathfrak{v}, \mathfrak{w}), \tau\right) \\ \upsilon\left(\Psi\left(\frac{\mathfrak{v}}{2}, \frac{\mathfrak{w}}{2}\right), \tau\right) \leq \upsilon'\left(\frac{\theta}{2}\Psi(\mathfrak{v}, \mathfrak{w}), \tau\right) \\ \varsigma\left(\Psi\left(\frac{\mathfrak{v}}{2}, \frac{\mathfrak{w}}{2}\right), \tau\right) \leq \varsigma'\left(\frac{\theta}{2}\Psi(\mathfrak{v}, \mathfrak{w}), \tau\right) \end{cases} \tag{5.1.1}$$

for some real θ with $0 < \theta < 1$, for all $\mathfrak{v}, \mathfrak{w} \in \mathcal{V}$, $\tau > 0$. Let $(\mathcal{W}, \eta, \upsilon, \varsigma)$ be a complete *NNS*. If $\xi : \mathcal{V} \to \mathcal{W}$ be a mapping satisfying (3.2.2). Then there exists a unique additive mapping $\mathfrak{Y} : \mathcal{V} \to \mathcal{W}$ defined by $\mathfrak{Y}(\mathfrak{v}) = (\eta, \upsilon, \varsigma) - \lim_{n\to\infty} 2^n \xi(\frac{\mathfrak{v}}{2^n})$ for all $\mathfrak{v} \in \mathcal{V}$, $\tau > 0$ satisfying

$$\begin{cases} \eta(\mathfrak{Y}(\mathfrak{v})-\xi(\mathfrak{v}),\tau) \geq \eta'\left(\Psi(\mathfrak{v},\mathfrak{v}),\frac{2(1-\theta)}{\theta}\tau\right), \\ \upsilon(\mathfrak{Y}(\mathfrak{v})-\xi(\mathfrak{v}),\tau) \leq \upsilon'\left(\Psi(\mathfrak{v},\mathfrak{v}),\frac{2(1-\theta)}{\theta}\tau\right), \\ \varsigma(\mathfrak{Y}(\mathfrak{v})-\xi(\mathfrak{v}),\tau) \leq \varsigma'\left(\Psi(\mathfrak{v},\mathfrak{v}),\frac{2(1-\theta)}{\theta}\tau\right). \end{cases} \tag{5.1.2}$$

Proof. Putting $\mathfrak{w} = \mathfrak{v}$ in (3.2.2) we get

$$\begin{cases} \eta(\xi(2\mathfrak{v})-2\xi(\mathfrak{v}),\tau) \geq \eta'(\Psi(\mathfrak{v},\mathfrak{v}),\tau), \\ \upsilon(\xi(2\mathfrak{v})-2\xi(\mathfrak{v}),\tau) \leq \upsilon'(\Psi(\mathfrak{v},\mathfrak{v}),\tau), \\ \varsigma(\xi(2\mathfrak{v})-2\xi(\mathfrak{v}),\tau) \leq \varsigma'(\Psi(\mathfrak{v},\mathfrak{v}),\tau). \end{cases} \tag{5.1.3}$$

Now, consider the set $\mathfrak{L} = \{\wp : \mathcal{V} \to \mathcal{W};\ \wp(0) = 0\}$ and introduce a complete generalized metric on $\mathfrak{L}$ as per Lemma (3.1). Define a mapping $\mathfrak{T} : \mathfrak{L} \to \mathfrak{L}$ by $\mathfrak{T}\wp(\mathfrak{v}) = 2\wp\left(\frac{\mathfrak{v}}{2}\right)$ for all $\wp \in \mathfrak{L}$ and $\mathfrak{v} \in \mathcal{V}$. As before it is easy to prove that $\mathfrak{T}$ is strictly contracting mapping with the Lipschitz constant $\theta < 1$. Also from (5.1.3),

$$\begin{cases} \eta\left(\xi(\mathfrak{v})-2\xi\left(\frac{\mathfrak{v}}{2}\right),\tau\right) \geq \eta'\left(\Psi\left(\frac{\mathfrak{v}}{2},\frac{\mathfrak{v}}{2}\right),\tau\right) \geq \eta'\left(\frac{\theta}{2}\Psi(\mathfrak{v},\mathfrak{v}),\tau\right) \\ \upsilon\left(\xi(\mathfrak{v})-2\xi\left(\frac{\mathfrak{v}}{2}\right),\tau\right) \leq \upsilon'\left(\Psi\left(\frac{\mathfrak{v}}{2},\frac{\mathfrak{v}}{2}\right),\tau\right) \leq \upsilon'\left(\frac{\theta}{2}\Psi(\mathfrak{v},\mathfrak{v}),\tau\right) \\ \varsigma\left(\xi(\mathfrak{v})-2\xi\left(\frac{\mathfrak{v}}{2}\right),\tau\right) \leq \varsigma'\left(\Psi\left(\frac{\mathfrak{v}}{2},\frac{\mathfrak{v}}{2}\right),\tau\right) \leq \varsigma'\left(\frac{\theta}{2}\Psi(\mathfrak{v},\mathfrak{v}),\tau\right). \end{cases}$$

Therefore $\mathfrak{d}(\xi,\mathfrak{T}\xi) \leq \frac{\theta}{2}$.

Again replacing $\mathfrak{v}$ by $2^{-(n+1)}\mathfrak{v}$ in (5.1.3) we get $\mathfrak{d}(\mathfrak{T}^{n+1}\xi,\mathfrak{T}^{n}\xi) \leq \frac{\theta^{n+1}}{2} < \infty$ as Lipschitz constant $\theta < 1$ for $n \geq m_0 = 1$.

Therefore, by Theorem (2.9) there exists a mapping $\mathfrak{Y} : \mathcal{V} \to \mathcal{W}$ fulfilling the following:

1. $\mathfrak{Y}$ is a fixed point of $\mathfrak{T}$, that is, $\mathfrak{Y}(\mathfrak{v}) = 2\mathfrak{Y}\left(\frac{\mathfrak{v}}{2}\right)$ for all $\mathfrak{v} \in \mathcal{V}$ and,

   $\mathfrak{d}(\xi,\mathfrak{Y}) \leq \{\mathfrak{d}(\xi,\mathfrak{T}\xi) + \mathfrak{d}(\mathfrak{T}\xi,\mathfrak{Y})\} < \infty$.
2. $\mathfrak{d}(\mathfrak{T}^{n}\xi,\mathfrak{Y}) =$

$$\inf\left\{\kappa \in \mathbb{R}^{+} : \begin{cases} \eta(\mathfrak{T}^{n}\xi(\mathfrak{v})-\mathfrak{Y}(\mathfrak{v}),\theta^{n}\kappa\tau) \geq \eta'(\Psi(\mathfrak{v},\mathfrak{v}),\tau), \\ \upsilon(\mathfrak{T}^{n}\xi(\mathfrak{v})-\mathfrak{Y}(\mathfrak{v}),\theta^{n}\kappa\tau) \leq \upsilon'(\Psi(\mathfrak{v},\mathfrak{v}),\tau), \text{ for all } \mathfrak{v} \in \mathcal{V}, \tau > 0 \\ \varsigma(\mathfrak{T}^{n}\xi(\mathfrak{v})-\mathfrak{Y}(\mathfrak{v}),\theta^{n}\kappa\tau) \leq \varsigma'(\Psi(\mathfrak{v},\mathfrak{v}),\tau). \end{cases}\right\}$$

.

Therefore $\mathfrak{d}(\mathfrak{T}^n\xi,\mathfrak{Y}) \le \theta^n\kappa \to 0$ as $n\to\infty$ and $\theta<1$. This implies the inequality

$$\mathfrak{Y}(\mathfrak{v}) = (\eta,\upsilon,\varsigma) - \lim_{n\to\infty}\mathfrak{T}^n\xi(\mathfrak{v}) = (\eta,\upsilon,\varsigma) - \lim_{n\to\infty} 2^n\xi\left(\frac{\mathfrak{v}}{2^n}\right) \quad \text{for all } \mathfrak{v}\in\mathcal{V}. \tag{5.1.4}$$

3. $\mathfrak{d}(\xi,\mathfrak{Y}) \le \frac{1}{1-\theta}\times\frac{\theta}{2} = \frac{\theta}{2(1-\theta)}$. It follows the results (5.1.2).

This completes the proof of the theorem.

Example 5.2. Let : $\mathcal{V}=\mathcal{W}=\mathfrak{Z}=\mathbb{R}$. Define $\xi:\mathbb{R}\to\mathbb{R}$ by $\xi(\mathfrak{v})=\mathfrak{v}$. Let $\eta(\mathfrak{v},\tau)=\eta'(\mathfrak{v},\tau)=e^{-\frac{|\mathfrak{v}|}{\tau}}$, $\upsilon(\mathfrak{v},\tau)=\upsilon'^{(\mathfrak{v},\tau)}=(1-e^{\frac{|\mathfrak{v}|}{\tau}})e^{-\frac{|\mathfrak{v}|}{\tau}}$ and $\varsigma(\mathfrak{v},\tau)=\varsigma'(\mathfrak{v},\tau)=(1-e^{\frac{|\mathfrak{v}|}{\tau}})$, for all $\mathfrak{v}\in\mathcal{V},\tau>0$. Let $\Psi(\mathfrak{v},\mathfrak{w})=\|\mathfrak{v}\|^2+\|\mathfrak{w}\|^2$. Clearly, the functions Ψ and ξ satisfies the conditions (5.1.1) and (3.2.2). Then, the unique additive function $\mathfrak{Y}(\mathfrak{v})=(\eta,\upsilon,\varsigma)-\lim_{n\to\infty}\frac{1}{2^n}\xi(\frac{\mathfrak{v}}{2^n})$, for all $\mathfrak{v}\in\mathcal{V},\tau>0$, exists that has satisfied the condition (4.5.2)

Corollary 5.3. Let $\mathcal{V}$ be a linear space with $\|.\|$, $q<1$ be a non-negative real number, $\mathfrak{z}_0\in\mathfrak{Z}$ and let $\xi:\mathcal{V}\to\mathcal{W}$ be a mapping satisfying (3.3.1) hold. Then there exists unique additive mapping $\mathfrak{Y}:\mathcal{V}\to\mathcal{W}$ for all $\mathfrak{v}\in\mathcal{V},\tau>0$ satisfying

$$\begin{cases}\eta(\mathfrak{Y}(\mathfrak{v})-\xi(\mathfrak{v}),\tau)\ge\eta'\left(\mathfrak{z}_0\|\mathfrak{v}\|^q,\frac{(2)^q-2}{2}\tau\right),\\ \upsilon(\mathfrak{Y}(\mathfrak{v})-\xi(\mathfrak{v}),\tau)\le\upsilon'\left(\mathfrak{z}_0\|\mathfrak{v}\|^q,\frac{(2)^q-2}{2}\tau\right),\\ \varsigma(\mathfrak{Y}(\mathfrak{v})-\xi(\mathfrak{v}),\tau)\le\varsigma'\left(\mathfrak{z}_0\|\mathfrak{v}\|^q,\frac{(2)^q-2}{2}\tau\right).\end{cases} \tag{5.3.1}$$

Proof. Define $\Psi(\mathfrak{v},\mathfrak{w})=\mathfrak{z}_0(\|\mathfrak{v}\|^q+\|\mathfrak{w}\|^q)$ and the proof derived from Theorem (5.1) by taking $\theta = 2^{1-q}$.

Theorem 5.4. Let $\mathcal{V}$ be linear space, $(\mathfrak{Z},\eta',\upsilon',\varsigma')$ be a neutrosophic normed space [NNS], $\Psi:\mathcal{V}\times\mathcal{V}\to\mathfrak{Z}$ be a function such that

$$\begin{cases}\eta(\Psi(\mathfrak{v},\mathfrak{w}),\tau)\ge\eta'\left(2\theta\Psi\left(\frac{\mathfrak{v}}{2},\frac{\mathfrak{w}}{2}\right),\tau\right),\\ \upsilon(\Psi(\mathfrak{v},\mathfrak{w}),\tau)\le\upsilon'\left(2\theta\Psi\left(\frac{\mathfrak{v}}{2},\frac{\mathfrak{w}}{2}\right),\tau\right),\\ \varsigma(\Psi(\mathfrak{v},\mathfrak{w}),\tau)\le\varsigma'\left(2\theta\Psi\left(\frac{\mathfrak{v}}{2},\frac{\mathfrak{w}}{2}\right),\tau\right).\end{cases} \tag{5.4.1}$$

for some real θ with $0 < \theta < 1$, for all $\mathfrak{v}, \mathfrak{w} \in \mathcal{V}$, $\tau > 0$. Let $(\mathcal{W}, \eta, \upsilon, \varsigma)$ be a complete *NNS*. If $\xi : \mathcal{V} \to \mathcal{W}$ be a mapping satisfying (3.2.2). Then there exists a unique additive mapping $\mathfrak{Y} : \mathcal{V} \to \mathcal{W}$ satisfying for all $\mathfrak{v} \in \mathcal{V}$, $\tau > 0$ satisfying

$$\begin{cases} \eta(\mathfrak{Y}(\mathfrak{v}) - \xi(\mathfrak{v}), \tau) \geq \eta'(\Psi(\mathfrak{v}, \mathfrak{v}), 2(1-\theta)\tau), \\ \upsilon(\mathfrak{Y}(\mathfrak{v}) - \xi(\mathfrak{v}), \tau) \leq \upsilon'(\Psi(\mathfrak{v}, \mathfrak{v}), 2(1-\theta)\tau), \\ \varsigma(\mathfrak{Y}(\mathfrak{v}) - \xi(\mathfrak{v}), \tau) \leq \varsigma'(\Psi(\mathfrak{v}, \mathfrak{v}), 2(1-\theta)\tau). \end{cases} \tag{5.4.2}$$

(for all $\mathfrak{v} \in \mathcal{V}$, $\tau > 0$)

Proof. Putting $\mathfrak{w} = \mathfrak{v}$ in (3.2.2) we get $\mathfrak{d}(\xi, \mathfrak{T}\xi) < \frac{1}{2}$.

As before we conclude the result (5.4.2).

Corollary 5.5. Let $\mathcal{V}$ be a linear space with $\|.\|$, $q > 1$ be a non-negative real number, $\mathfrak{z}_0 \in \mathfrak{Z}$ and let $\xi : \mathcal{V} \to \mathcal{W}$ be a mapping satisfying (3.3.1) hold. Then there exists unique additive mapping $\mathfrak{Y} : \mathcal{V} \to \mathcal{W}$ for all $\mathfrak{v} \in \mathcal{V}, \tau > 0$ satisfying

$$\begin{cases} \eta(\mathfrak{Y}(\mathfrak{v}) - \xi(\mathfrak{v}), \tau) \geq \eta'\left(\mathfrak{z}_0 \|\mathfrak{v}\|^q, \frac{2-(2)^q}{2}\tau\right), \\ \upsilon(\mathfrak{Y}(\mathfrak{v}) - \xi(\mathfrak{v}), \tau) \leq \upsilon'\left(\mathfrak{z}_0 \|\mathfrak{v}\|^q, \frac{(2-(2)^q}{2}\tau\right), \\ \varsigma(\mathfrak{Y}(\mathfrak{v}) - \xi(\mathfrak{v}), \tau) \leq \varsigma'\left(\mathfrak{z}_0 \|\mathfrak{v}\|^q, \frac{2-(2)^q}{2}\tau\right). \end{cases} \tag{5.5.1}$$

Proof. Define $\Psi(\mathfrak{v}, \mathfrak{w}) = \mathfrak{z}_0(\|\mathfrak{v}\|^q + \|\mathfrak{w}\|^q)$ and the proof derived from Theorem (5.4) by taking $\theta = 2^{q-1}$.

Theorem 5.6. Let $\mathcal{V}$ be linear space, $(\mathfrak{Z}, \eta', \upsilon', \varsigma')$ be a neutrosophic normed space [*NNS*], $\Psi : \mathcal{V} \times \mathcal{V} \to \mathfrak{Z}$ be a function such that

$$\begin{cases} \eta\left(\Psi\left(\frac{\mathfrak{v}}{2}, \frac{\mathfrak{w}}{2}\right), \tau\right) \geq \eta'\left(\frac{\theta}{2}\Psi\left(\frac{\mathfrak{v}}{2}, \frac{\mathfrak{w}}{2}\right), \tau\right), \\ \upsilon\left(\Psi\left(\frac{\mathfrak{v}}{2}, \frac{\mathfrak{w}}{2}\right), \tau\right) \leq \upsilon'\left(\frac{\theta}{2}\Psi\left(\frac{\mathfrak{v}}{2}, \frac{\mathfrak{w}}{2}\right), \tau\right), \\ \varsigma\left(\Psi\left(\frac{\mathfrak{v}}{2}, \frac{\mathfrak{w}}{2}\right), \tau\right) \leq \varsigma'\left(\frac{\theta}{2}\Psi\left(\frac{\mathfrak{v}}{2}, \frac{\mathfrak{w}}{2}\right), \tau\right). \end{cases} \tag{5.6.1}$$

for some real θ with $0 < \theta < 1$, for all $\mathfrak{v}, \mathfrak{w} \in \mathcal{V}$, $\tau > 0$. Let $(\mathcal{W}, \eta, \upsilon, \varsigma)$ be a complete NNS. If $\xi : \mathcal{V} \to \mathcal{W}$ be a mapping with $\xi(0) = 0$ satisfying (4.1.1). Then there exists a unique additive mapping $\mathfrak{Y} : \mathcal{V} \to \mathcal{W}$ satisfying

$$\begin{cases} \eta(\mathfrak{Y}(\mathfrak{v})-\xi(\mathfrak{v}),\tau) \geq \eta'(\Psi(\mathfrak{v},0),(1-\theta)\tau) \\ \upsilon(\mathfrak{Y}(\mathfrak{v})-\xi(\mathfrak{v}),\tau) \leq \upsilon'(\Psi(\mathfrak{v},0),(1-\theta)\tau) \\ \varsigma(\mathfrak{Y}(\mathfrak{v})-\xi(\mathfrak{v}),\tau) \leq \varsigma'(\Psi(\mathfrak{v},0),(1-\theta)\tau) \end{cases} \quad (5.6.2)$$

(for all $\mathfrak{v} \in \mathcal{V}, \tau > 0$)

Proof. Putting $\mathfrak{w} = 0$ in (4.1.1) and the rest of the proof of the theorem similar as before.

Corollary 5.7. Let $\mathcal{V}$ be a linear space with norm $\|.\|$, $q < 1$ be a non-negative real number, $\mathfrak{z}_0 \in \mathfrak{Z}$ and let $\xi : \mathcal{V} \to \mathcal{W}$ be a mapping with $\xi(0) = 0$ such that (4.1.3) hold. Then there exists unique additive mapping $\mathfrak{Y} : \mathcal{V} \to \mathcal{W}$ for all $\mathfrak{v} \in \mathcal{V}, \tau > 0$ satisfying

$$\begin{cases} \eta(\mathfrak{Y}(\mathfrak{v})-\xi(\mathfrak{v}),\tau) \geq \eta'(\mathfrak{z}_0\|\mathfrak{v}\|^q, \frac{(2)^q-2}{2}\tau) \\ \upsilon(\mathfrak{Y}(\mathfrak{v})-\xi(\mathfrak{v}),\tau) \leq \upsilon'(\mathfrak{z}_0\|\mathfrak{v}\|^q, \frac{(2)^q-2}{2}\tau) \\ \varsigma(\mathfrak{Y}(\mathfrak{v})-\xi(\mathfrak{v}),\tau) \leq \varsigma'(\mathfrak{z}_0\|\mathfrak{v}\|^q, \frac{(2)^q-2}{2}\tau) \end{cases} \quad (5.6.3)$$

Proof. Define $\Psi(\mathfrak{v},\mathfrak{w}) = \mathfrak{z}_0(\|\mathfrak{v}\|^q + \|\mathfrak{w}\|^q)$ and the proof derived from Theorem (5.6) by taking $\theta = 2^{1-q}$.

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