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EXISTENCE OF SOLUTION OF INTEGRAL EQUATIONS IN CONE METRIC SPACES

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Abstract

In this study, we use comparison mapping for contractive conditions in cone metric spaces to demonstrate the existence of a solution to mixed type integral equations. We use examples to show how our findings can be put to use.

Keywords: Integral equation, Cone metric space, Contractive mapping.

MSC (2020): 47H10, 54E50, 47G20, 34K20

1. Introduction

Fractional The French mathematician Frechet [15] originally described and investigated an abstract metric space in 1906. Numerous researchers [6, 8, 10, 18, 19, 20] have expanded the definition of metric space to include cone, semi, and quasi metric spaces, as well as generalized contraction mappings with applications.

One of the most vital areas of mathematics is fixed point theory, with numerous applications in several aspects of physical science as well as various domains of mathematics. The Banach contraction principle has a large number of generalizations like Ciric contraction, S-contraction, Chatterjee contraction, weak contractions principle etc. These contractions proved many results in analysis and play a key part in determining the existence and uniqueness of the findings in analysis and many other areas of mathematics.

Branciari [3] derived results in a fixed point theory for a single mapping in 2002. The authors [1, 2, 9, 11, 12, 16] used integral type equations to demonstrate several fixed point theorems.

2. Preliminaries

Definition 2.1 [7, 13, 14] Assume that B is a real Banach space and consider a subset $\mathbb{C}$ of B is referred to as a cone if satisfy:

1. $\mathbb{C} \neq \{0\}$, nonempty and closed
2. $a, b \in R^{+},\ \alpha, \beta \in \mathbb{C} \ \Rightarrow a\alpha + b\beta \in \mathbb{C}$;
3. $\alpha \in \mathbb{C}$ and $-\alpha \in \mathbb{C} \Rightarrow \alpha = 0$.

A partial order relation $\leqslant$ on a cone $\mathbb{C}$ define as $\alpha \leqslant \beta$ if and only if $\beta - \alpha \in \mathbb{C}$. If there is a positive number $a > 0$ such that

$$0 \leqslant \alpha \leqslant \beta \Rightarrow \| \alpha \| \leqslant a \| \beta \|,$$

for all $\alpha, \beta \in B$, then cone $\mathbb{C}$ is known as normal and the least positive value which holds the above inequality is known as normal constant of cone $\mathbb{C}$.

Definition 2.2 [7, 13, 14] Consider B be a Banach space and X be a nonempty subset of B, then a mapping $d: X \times X \to B$ is known as cone and (X, d) is known as cone metric space if satisfies:

(i) $d(\alpha, \beta) \geq 0$ and $d(\alpha, \beta) = 0$ if and only if $\alpha = \beta$;

(ii) $d(\alpha, \beta) = d(\beta, \alpha)$;

(iii) $d(\alpha, \beta) \leq d(\alpha, \gamma) + d(\gamma, \beta)$

for all $\alpha, \beta, \gamma \in X$.

Example 2.1 [13, 17] Consider the supremum norm on the the Banach space $\mathrm{B} = \mathbb{C}([0, \mathrm{l}], \mathbb{Z})$, which are all the continuous functions from $[0, \mathrm{l}]$ into $\mathbb{Z}$, such that

$$\| \alpha \|_{\infty} = sup\{\| \alpha(s) \| : s \in [0, l]\}.$$

Assume $\mathbb{C} = \{(\alpha, \beta) : \alpha, \beta \geq 0\} \subset B = \mathbb{R}^2$ and $d(\zeta, \xi) = (\| \zeta - \xi \|, \kappa \| \zeta - \xi \|_{\infty})$, for every $\zeta, \xi \in B$. Then (B, d) is a cone metric space on the supremum norm.

Definition 2.3 [17] Consider a function $\Psi: X \to X$ on an ordered space X is sid to be comparison function if it implies $\alpha \leq \beta$, $\Psi(\alpha) \leq \Psi(\beta)$, $\Psi(\alpha) \leq \alpha$ *and* $lim_{n\to\infty} \parallel \Psi^n(\alpha) \parallel = 0$, for every $\alpha, \beta \in X$.

Example 2.2 [17] Let $\mathbb{C} = \{(\alpha, \beta) \in B: \alpha, \beta \geq 0\}$, where $B = \mathbb{R}^2$, then a mapping $\Psi: B \to B$ defined as $\Psi(\alpha, \beta) = (a\alpha, a\beta)$ is a comparison function for $a \in (0,1)$. If Ψ_1, Ψ_2 are two comparison functions over $\mathbb{R}$, then $\Psi(\alpha, \beta) = (\Psi_1(\alpha), \Psi_2(\beta))$ is also a comparison function over B.

Lemma 2.1 [17] Given that normal constant β on a normal cone $\mathbb{C}$. Let $f: X \to X$ be a function on a complete cone metric space (X, d) as follows

$$d(f(\alpha), f(\beta)) \leq \Psi(d(\alpha, \beta)),$$

for all $\alpha, \beta \in X$. If it is exists as a comparison function $\Psi: \mathbb{C} \to \mathbb{C}$, then f has a unique fixed point.

The aim of the current research is to generalized the findings of Pachpatte [4, 5] and numerous other articles by investigating the existence and uniqueness of solutions to the integrodifferential equations using fixed point theory on the cone metric spaces.

3. Main Results

Theorem 3.1 Assume the integrodifferential equations on a complete cone metric space (X, d) as follows

$$u(s) = f(s) + \int_0^s \zeta\big(s, t, u(t)\big)dt + \int_0^l \xi\big(s, t, u(t)\big)dt, \quad s \in [0, l], \tag{3.1}$$

$$u'(s) = f(s) + \int_0^s \zeta\big(s, t, u(t)\big)dt + \int_0^l \xi\big(s, t, u(t)\big)dt, \quad u(0) = u_0, \tag{3.2}$$

where $f: [0, l] \to \mathbb{Z}$, $\zeta, \xi: [0, l] \times [0, l] \times \mathbb{Z} \to \mathbb{Z}$ are continuous functions and $\mathbb{Z}$ is a Banach space on the norm $\parallel . \parallel$, provided $u_0 \in \mathbb{Z}$, If the following criteria are met by integrodifferential equations:

$(\mathbb{C}_1)$ A comparison function $\Psi: \mathbb{R}^+ \to \mathbb{R}^+$ and continuous functions $q_1, q_2: [0, l] \times [0, l] \to \mathbb{R}^+$ are exists, such that

$$(\parallel \zeta(s, t, p) - \zeta(s, t, q) \parallel, \beta \parallel \zeta(s, t, p) - \zeta(s, t, q) \parallel) \leq q_1(s, t)\Psi(d(p, q)),$$

$$(\parallel \xi(s, t, p) - \xi(s, t, q) \parallel, \beta \parallel \xi(s, t, p) - \xi(s, t, q) \parallel) \leq q_2(s, t)\Psi(d(p, q)),$$

for every $s, t \in [0, l]$ and $p, q \in \mathbb{Z}$.

$$(\mathbb{C}_2) \quad \sup_{s\in[0,l]} \int_0^l [q_1(s,t) + q_2(s,t)]dt = 1, \int_0^l \int_0^l [q_1(s,t) + q_2(s,t)]dt \;\; dt \leq 1.$$

Then the unique solution of integrodifferential equations (3.1) and (3.2) are exists on $[0, l]$.

Proof : Consider the operator $\Gamma: B \to B$ is

$$\Gamma u(s) = f(s) + \int_0^s \zeta\big(s,t,u(t)\big)dt + \int_0^l \xi\big(s,t,u(t)\big)dt, \quad s \in [0,l]. \qquad (3.3)$$

By conditions $\mathbb{C}_1$ and $\mathbb{C}_2$, we have

$$\begin{aligned}
& (\parallel \Gamma u(s) - \Gamma v(s) \parallel, \beta \parallel \Gamma u(s) - \Gamma v(s) \parallel) \\
\leq \;& (\parallel \int_0^s \zeta(s,t,u(t))dt + \int_0^l \xi(s,t,u(t))dt - \int_0^s \zeta(s,t,v(t))dt - \int_0^l \xi(s,t,v(t))dt \parallel, \\
& \beta \parallel \int_0^s \zeta(s,t,u(t))dt + \int_0^l \xi(s,t,u(t))dt - \int_0^s \zeta(s,t,v(t))dt - \int_0^l \xi(s,t,v(t))dt \parallel) \\
\leq \;& (\int_0^s \parallel \zeta(s,t,u(t)) - \zeta(s,t,v(t)) \parallel dt + \int_0^l \parallel \xi(s,t,u(t)) - \xi(s,t,v(t)) \parallel dt, \\
& \beta \int_0^s \parallel \zeta(s,t,u(t)) - \zeta(s,t,v(t)) \parallel dt + \beta \int_0^l \parallel \xi(s,t,u(t)) - \xi(s,t,v(t)) \parallel dt) \\
\leq \;& (\int_0^s \parallel \zeta(s,t,u(t)) - \zeta(s,t,v(t)) \parallel dt, \beta \int_0^s \parallel \zeta(s,t,u(t)) - \zeta(s,t,v(t)) \parallel dt) \\
& +(\int_0^l \parallel \xi(s,t,u(t)) - \xi(s,t,v(t)) \parallel dt, \beta \int_0^l \xi(s,t,u(t)) - \xi(s,t,v(t)) \parallel dt) \\
\leq \;& \int_0^s q_1(s,t)\Psi(\parallel u - v \parallel_\infty, \beta \parallel u - v \parallel_\infty)dt + \int_0^l q_2(s,t)\Psi(\parallel u - v \parallel_\infty, \beta \parallel u - v \parallel_\infty)dt \\
\leq \;& \int_0^l q_1(s,t)\Psi(\parallel u - v \parallel_\infty, \beta \parallel u - v \parallel_\infty)dt + \int_0^l q_2(s,t)\Psi(\parallel u - v \parallel_\infty, \alpha \parallel u - v \parallel_\infty)dt \\
\leq \;& \int_0^l [q_1(s,t) + q_2(s,t)]\Psi(\parallel u - v \parallel_\infty, \beta \parallel u - v \parallel_\infty)dt \\
\leq \;& \Psi(\parallel u - v \parallel_\infty, \beta \parallel u - v \parallel_\infty) \int_0^l [q_1(s,t) + q_2(s,t)]dt \\
= \;& \Psi(\parallel u - v \parallel_\infty, \beta \parallel u - v \parallel_\infty),
\end{aligned}$$

This means that

$$d(\Gamma u, \Gamma v) \leq \Psi(d(u,v))$$

for every $u, v \in B$. Now, the Lemma 2.1 has been applied, therefore the operator Γ has a unique fixed point in B and hence a unique solution of equations (3.1) and (3.2). Thus Theorem 3.1 has proved.

Example 3.1 Consider the mappings ζ and ξ such that

$$\zeta(s,t,u) = st + \frac{ut}{2},\ h(s,t,u) = (st)^2 + \frac{stu^2}{2},\ s,t \in [0,1]\ u \in \mathbb{C}([0,1],\mathbb{R}), \quad (3.4)$$

and a metric $d(u,v)$ described as

$$d(u,v) = (\parallel u - v \parallel_\infty, \alpha \parallel u - v \parallel_\infty)$$

on $\mathbb{C}([0,1],\mathbb{R})$ and $\beta \geq 0$. Then $\mathbb{C}([0,1],\mathbb{R})$ is a complete cone metric space. We choose two continuous functions $q_1^*(s,t)$, $q_2^*(s,t)$: $[0,1] \times [0,1] \to \mathbb{R}^+$, such that $q_1^*(s,t) = t$ and $q_2^*(s,t) = st$ and a comparison function $\Psi^*: \mathbb{R}^2 \to \mathbb{R}^2$ such that $\Psi^*(u,v) = \frac{1}{2}(u,v)$. Now, we demonstrate that

$$\begin{aligned}
(|\zeta(s, \quad & t, u(t)) - \zeta(s,t,v(t))|, \beta|\zeta(s,t,u(t)) - \zeta(s,t,v(t))|) \\
&= (|st + \frac{ut}{2} - st - \frac{vt}{2}|, \beta|st + \frac{ut}{2} - st - \frac{vt}{2}|) \\
&= (|\frac{ut}{2} - \frac{vt}{2}|, \beta|\frac{ut}{2} - \frac{vt}{2}|) \\
&= \frac{t}{2}(|u - v|, \beta|u - v|) \\
&\leq \frac{t}{2}(\parallel u - v \parallel_\infty, \beta \parallel u - v \parallel_\infty) \\
&= q_1^* \Psi^*(\parallel u - v \parallel_\infty, \beta \parallel u - v \parallel_\infty),
\end{aligned}$$

Similarly, we can show that

$$\left(\left|\xi\left(s,t,u(t)\right) - \xi\left(s,t,v(t)\right)\right|, \beta\left|\xi\left(s,t,u(t)\right) - \xi\left(s,t,v(t)\right)\right|\right) \leq q_2^* \Psi^*(\parallel u - v \parallel_\infty, \beta \parallel u - v \parallel_\infty),$$

Now, we obtain

$$\int_0^1 [q_1^*(s,t) + q_2^*(s,t)]dt = \int_0^1 [t + st]dt = \frac{1}{2}(1+s) \Rightarrow \sup_{s\in[0,1]} \{\frac{1}{2}(1+s)\} = 1.$$

$$\int_0^1 \int_0^1 [q_1^*(s,t) + q_2^*(s,t)]dtds = \int_0^1 \int_0^1 [t + st]dtds = \int_0^1 \frac{1}{2}(1+s)ds \leq \frac{3}{4} < 1.$$

Thus fulfilled each requirement of Theorem 3.1, therefore the validity of solution of the integrodifferential equations are existence and uniqueness.

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