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WEAK GALERKIN FINITE ELEMENT METHODS FOR GRAD-DIV ELLIPTIC PROBLEMS

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Abstract

In this work, we present a weak Galerkin (WG) finite element scheme for grad-div elliptic problems with stabilizers. Optimal orders of convergence are established for the WG approximations in discrete energy norm. The WG method as applied to grad-div problem uses discrete weak divergence with appropriately defined stabilizations that enforce a weak continuity of the approximating functions. A numerical test is presented to demonstrate the effectiveness of the proposed method.

Keywords: $\mathbf{H}(\mathrm{div};\Omega)$-elliptic problems, weak Galerkin finite element method, weak divergence, polygonal mesh.

2020 AMS Classifications: 35J47, 65N15, 65N30.

1. Introduction

In this work, we consider a grad-div elliptic problem of the form

$$-\nabla(\alpha\nabla\cdot\mathbf{u})+\beta\mathbf{u} = \mathbf{f} \text{ in } \Omega \quad (1.1)$$

$$\mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega \quad (1.2)$$

where $\Omega\subset\mathbb{R}^2$ is a convex polygonal domain with boundary $\partial\Omega$ and $\mathbf{n}$ stands for outward unit normal vector to the boundary $\partial\Omega$. Further, we assume that α and β are uniformly positive coefficients in $L^\infty(\Omega)$, and $\mathbf{f}:\Omega\subset\mathbb{R}^2\rightarrow[L^2(\Omega)]^2$ is the given source function. The above system of equations is also known as $\mathbf{H}(\text{div};\Omega)$-elliptic problems.

The significance of $\mathbf{H}(\text{div};\Omega)$-elliptic problems have caused robust research into coherent numerical schemes. Numerical approximations for $\mathbf{H}(\text{div};\Omega)$ elliptic problems have been studied extensively in existing literature (cf. [2, 3,6,7,8,10,12] to name a few). Recently, the newly introduced weak Galerkin finite element method (WG-FEM) (cf. [13]) has attracted much attention in the field of numerical partial differential equations. The WG finite element approximations are derived from weak formulations of the problems by replacing the involved differential operators by its weak forms and adding parameter free stabilizers. In fact, WG formulation is a natural extension of conforming finite element formulation when nonconforming elements are used. The concept of weak derivatives makes WG a widely applicable numerical technique for a large variety of PDEs arising from the mathematical modeling of practical problems in science and engineering. We refer to [13,14] for full scale study of theory and algorithm of WG-FEMs. Present work deals with the convergence analysis of stabilizer based WG-FEM for $\mathbf{H}(\text{div};\Omega)$-elliptic problems on the WG finite element space $([\mathcal{P}_k(T^0)]^2,[\mathcal{P}_{k-1}(\partial T)]^2,\mathcal{P}_{k-1}(T))$, where $k\geq 1$ is the degree of polynomials in the interior of the element T. A comparative study on weak Galerkin finite element methods (WGFEMs) with the widely accepted discontinuous Galerkin finite element methods (DGFEMs) and the classical mixed finite element methods (MFEMs) can be found in [9].

In this paper, we will use standard notation for Sobolev spaces and norms (cf. [1]). For any domain $\mathcal{D}\subset\mathbb{R}^2$ and integer $l\geq 0, H^l(\mathcal{D})$ denotes the standard Sobolev space of order l equipped with the norm $\|\cdot\|_{l,\mathcal{D}}$ and inner product $(\cdot,\cdot)_{l,\mathcal{D}}$. The space $H^0(\mathcal{D})$ coincides with $L^2(\mathcal{U})$, for which the norm and the inner product are denoted by $\|\cdot\|_{\mathcal{D}}$ and $(\cdot,\cdot)_{\mathcal{D}}$, respectively. For our convenience, we remove the subscript $\mathcal{D}$ in the norm and inner product notation when $\mathcal{D}=\Omega$. In addition, we stick to following usual vector valued Sobolev spaces (see, Chap. 1 in [5] or [11])

$$\begin{aligned} \mathbf{H}(\mathrm{div};\Omega) &= \{\mathbf{v}:\mathbf{v}\in[L^2(\Omega)]^2,\nabla\cdot\mathbf{v}\in L^2(\Omega)\},\\ \mathbf{H_0}(\mathrm{div};\Omega) &= \{\mathbf{v}\in\mathbf{H}(\mathrm{div};\Omega),\mathbf{v}\cdot\mathbf{n}=0 \text{ on } \partial\Omega\},\\ \mathbf{H^1}(\mathrm{div};\Omega) &= \left\{\mathbf{v}\in\mathbf{H^1}(\Omega),\nabla\cdot\mathbf{v}\in H^1(\Omega)\right\}, \end{aligned}$$

where $\mathbf{H}^1(\Omega)=[H^1(\Omega)]^2$. In general, $\mathbf{H}^l(\mathcal{D})=[H^l(\mathcal{D})]^2$ denotes the vector valued Hilbertian Sobolev spaces.

We end this section with the variational formulation of (1.1) - (1.2): Find $\mathbf{u}\in\mathbf{H_0}(\mathrm{div};\Omega)$ such that

$$\int_\Omega \alpha\mathrm{div}\ \mathbf{u}\cdot\mathrm{div}\ \mathbf{v}dx+\int_\Omega \beta\mathbf{u}\cdot\mathbf{v}dx=\int_\Omega \mathbf{f}\cdot\mathbf{v}dx\ \forall\mathbf{v}\in\mathbf{H_0}(\mathrm{div};\Omega). \tag{1.3}$$

Existence and uniqueness of the solution of (1.3) is ensured by the Lax-Milgram Lemma [4].

2. Weak Galerkin Spaces and Schemes

This section deals with the weak Galerkin finite element discretization for the problem (1.1) - (1.2) and introduces the definition of the weak divergence operator.

Consider a partition $\mathcal{T}_h$ of the domain Ω consisting of polygons and satisfying a set of conditions specified in [14]. Denote by $\mathcal{E}_h$ the set of all edges in $\mathcal{T}_h$ and let $\mathcal{E}_h^0=\mathcal{E}_h\setminus\partial\Omega$ be the set of all interior edges. For every element $T\in\mathcal{T}_h$, we denote by $|T|$ the measure of T and by h_T its diameter, and mesh size $h=\max_{T\in\mathcal{T}_h}h_T$.

Let T be any polygonal domain with interior T^0 and boundary ∂T. A weak function on the region T refers to a pair of vector valued functions $\mathbf{v}=\{\mathbf{v}_0,\mathbf{v}_b\}$ such that $\mathbf{v}_0\in[L^2(T)]^2$ and $\mathbf{v}_b\cdot\boldsymbol{\eta}\in L^2(\partial T)$, where $\boldsymbol{\eta}$ is the unit outward normal direction to ∂T. We now introduce following weak Galerkin space

$$\Sigma_h=\{\boldsymbol{\sigma}_h=\{\boldsymbol{\sigma}_0,\boldsymbol{\sigma}_b\}:\{\boldsymbol{\sigma}_0,\boldsymbol{\sigma}_b\}|_T\in[\mathcal{P}_k(T^0)]^2\times[\mathcal{P}_{k-1}(e)]^2,e\subset\partial T,T\in\mathcal{T}_h\} \tag{2.1}$$

For the well-possed weak Galerkin approximation, we would like to emphasize that there is only a single value $\boldsymbol{\sigma}_b$ defined on each edge $e \in \mathcal{E}_h$ (cf. [14, 15]). Accordingly, we define following discrete weak Galerkin space

$$\mathbf{V}_h = \left\{ \boldsymbol{\sigma}_h = \{\boldsymbol{\sigma}_0, \boldsymbol{\sigma}_b\} \in \sum_h : [\boldsymbol{\sigma}_b]_e = \mathbf{0} \forall e \in \mathcal{E}_h^0 \right\}.$$

Here, $[\cdot]_e$ denotes the jump across an interior edge $e \in \mathcal{E}_h^0$. Subsequently, we define

$$\mathbf{V}_h^0 = \{\boldsymbol{\sigma}_h = \{\boldsymbol{\sigma}_0, \boldsymbol{\sigma}_b\} \in \mathbf{V}_h : \boldsymbol{\sigma}_b \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$$

Next, we proceed to define the discrete weak divergence operator. For any $\mathbf{v}_h = \{\mathbf{v}_0, \mathbf{v}_b\} \in \sum_h$, the discrete weak divergence operator, denoted by $\nabla_w \cdot \mathbf{v}_h$, defined as the unique polynomial $\nabla_w \cdot \mathbf{v}_h \in \mathcal{P}_{k-1}(T)$ that satisfies the following equation

$$(\nabla_w \cdot \mathbf{v}_h, \varphi)_T = -\int_T \mathbf{v}_0 \cdot (\nabla\varphi) dT + \int_{\partial T} \mathbf{v}_b \cdot \boldsymbol{\eta} \varphi ds \; \forall \varphi \in \mathcal{P}_{k-1}(T). \quad (2.2)$$

For each element $T \in \mathcal{T}_h$, denote by $\mathbf{Q}_0^k$ the usual L^2 projection operator from $[L^2(T)]^2$ onto $[\mathcal{P}_k(T)]^2$. For each edge $e \in \mathcal{E}_h$, denote by $\mathbf{Q}_b^j$ the L^2 projection operator from $[L^2(e)]^2$ onto $[\mathcal{P}_j(e)]^2$. For $\mathbf{u} \in \mathbf{H}^1(\text{div}; \Omega)$, we shall combine $\mathbf{Q}_0^k$ with $\mathbf{Q}_b^j$ by writing $\mathbf{Q}_h \mathbf{u} = \{\mathbf{Q}_0^k \mathbf{u}, \mathbf{Q}_b^j \mathbf{u}\}$. Apart $\mathbf{Q}_h$ projection, let $\mathbb{Q}_h^r$ be the usual L^2 projection operator from $L^2(T)$ onto $\mathcal{P}_r(T), r \geq 0$. For both projection operators, the following identity holds (cf. [15])

$$\mathbb{Q}_h^{k-1}(\nabla \cdot \mathbf{v}) = \nabla_w \cdot (\mathbf{Q}_h \mathbf{v}) \forall \mathbf{v} \in \mathbf{H}^1(\text{div}; \Omega), \quad (2.3)$$

where $\mathbf{Q}_h \mathbf{v} = \{\mathbf{Q}_0^k \mathbf{v}, \mathbf{Q}_b^{k-1} \mathbf{v}\}$.

Now, we recall following crucial approximation properties for local projections $\mathbf{Q}_0^k$ and $\mathbb{Q}_h^r$. For details, we refer to ([14,15]).

Lemma 2.1. Let $\mathcal{T}_h$ be a finite element partition of Ω satisfying the shape regularity assumption as specified in [14]. Then, we have

$$\sum_{T\in\mathcal{T}_h} \|\mathbf{w}-\mathbf{Q}_0^k\mathbf{w}\|_T^2 \le Ch^{2(s+1)} \|\mathbf{w}\|_{s+1}^2,\ 0\le s\le k,$$
$$\sum_{T\in\mathcal{T}_h} \|\nabla(\mathbf{w}-\mathbf{Q}_0^k\mathbf{w})\|_T^2 \le Ch^{2s} \|\mathbf{w}\|_{s+1}^2,\ 0\le s\le k,$$
$$\sum_{T\in\mathcal{T}_h} \left\{\|z-\mathbb{Q}_h^r z\|_T^2 + h_T^2\|\nabla(z-\mathbb{Q}_h^r z)\|_T^2\right\} \le Ch^{2(s+1)} \|z\|_{s+1}^2,\ 0\le s\le r.$$

Let T be an element with e as an edge. For any function $\varphi\in H^1(T)$, the following trace inequality holds true (see, [14] for details)

$$\|\varphi\|_e^2 \le C(h_T^{-1}\|\varphi\|_T^2 + h_T\|\nabla\varphi\|_T^2). \tag{2.4}$$

For any piecewise polynomial φ of degree p on $\mathcal{T}_h$, there exists constant $C=C(p)$ such that (cf. [14])

$$\|\nabla\varphi\|_T \le C(p)h_T^{-1}\|\varphi\|_T \quad \forall T\in\mathcal{T}_h \tag{2.5}$$

3. Error Analysis for the WG-FEM with Stabilizer

Here, we propose a WG-FEM with stabilizer for the problem (1.1)-(1.2) based on the local weak Galerkin space $(\mathcal{P}_k^2, \mathcal{P}_{k-1}^2, \mathcal{P}_{k-1})$.

Now, we introduce a bilinear map $\mathcal{A}_1: \mathbf{V}_h\times\mathbf{V}_h\to\mathbb{R}$ to be used in this section as follows

$$\mathcal{A}_1(\mathbf{u}_h,\mathbf{v}_h) = (\alpha\nabla_w\cdot\mathbf{u}_h, \nabla_w\cdot\mathbf{v}_h) + (\beta\mathbf{u}_0,\mathbf{v}_0) + \mathcal{S}(\mathbf{u}_h,\mathbf{v}_h), \tag{3.1}$$

where $\nabla_w.$ is the discrete weak divergence operator as defined in (2.2) and the stabilizer $\mathcal{S}: \mathbf{V}_h\times\mathbf{V}_h\to\mathbb{R}$ is defined as

$$\mathcal{S}(\mathbf{u}_h,\mathbf{v}_h) = \sum_{T\in\mathcal{T}_h} h_T^{-1}\langle \mathbf{Q}_b^{k-1}\mathbf{u}_0-\mathbf{u}_b, \mathbf{Q}_b^{k-1}\mathbf{v}_0-\mathbf{v}_b\rangle_{\partial T}. \tag{3.2}$$

Here, $\langle\cdot,\cdot\rangle_{\partial T}$ denotes the L^2 inner product on ∂T and we write

$$\langle\cdot,\cdot\rangle_{\partial T} = \sum_{e\subset\partial T} \langle\cdot,\cdot\rangle_e$$

where $\langle\cdot,\cdot\rangle_e$ denotes the L^2 inner product on $e \in \mathcal{E}_h$.

Then, it is easy to verify that the weak finite element space $\mathbf{V}_h^0$ is a normed linear space with respect to following triple-bar norm given by

$$\begin{aligned} \|\mathbf{v}_h\|_1^2 \quad &:= \textstyle\sum_{T\in\mathcal{T}_h} \left\|\alpha^{\frac{1}{2}}\nabla_w\cdot\mathbf{v}_h\right\|_T^2 + \left\|\beta^{\frac{1}{2}}\mathbf{v}_0\right\|_T^2 + \sum_{T\in\mathcal{T}_h} h_T^{-1}\|\mathbf{Q}_b^{k-1}\mathbf{v}_0-\mathbf{v}_b\|_{\partial T}^2 \\ &= \mathcal{A}_1(\mathbf{v}_h,\mathbf{v}_h),\ \mathbf{v}_h=\{\mathbf{v}_0,\mathbf{v}_b\}\in\mathbf{V}_h^0. \end{aligned} \tag{3.3}$$

Weak Galerkin Algorithm 1 (WGALG-1). A numerical approximation for (1.1)-(1.2) can be obtained by seeking $\mathbf{u}_h=\{\mathbf{u}_0,\mathbf{u}_b\}\in\mathbf{V}_h^0$ such that

$$\mathcal{A}_1(\mathbf{u}_h,\mathbf{v}_h) = (\mathbf{f},\mathbf{v}_0)\forall\mathbf{v}_h=\{\mathbf{v}_0,\mathbf{v}_b\}\in\mathbf{V}_h^0. \tag{3.4}$$

The well-posedness for the problem (3.4) follows from the fact that $\mathbf{V}_h^0$ is a normed linear space with respect to the triple-bar norm.

Before proceeding further, we derive following results for our later analysis.

Lemma 3.1. For any $\mathbf{v}_h=\{\mathbf{v}_0,\mathbf{v}_b\}\in\mathbf{V}_h$, we have

$$\textstyle\sum_{T\in\mathcal{T}_h} \|\nabla\cdot\mathbf{v}_0\|_T^2 \le C\|\mathbf{v}_h\|_1^2 \tag{3.5}$$

Proof. For $\mathbf{v}_h=\{\mathbf{v}_0,\mathbf{v}_b\}\in\mathbf{V}_h$, using the integration by parts and weak divergence operator, we obtain

$$\begin{aligned} (\nabla\cdot\mathbf{v}_0,\nabla\cdot\mathbf{v}_0)_T &= -\big(\mathbf{v}_0,\nabla(\nabla\cdot\mathbf{v}_0)\big)_T + \langle\mathbf{v}_0\cdot\boldsymbol{\eta},\nabla\cdot\mathbf{v}_0\rangle_{\partial T} \\ &= -\big(\mathbf{v}_0,\nabla(\nabla\cdot\mathbf{v}_0)\big)_T + \langle\mathbf{v}_b\cdot\boldsymbol{\eta},\nabla\cdot\mathbf{v}_0\rangle_{\partial T} \\ &\quad +\langle(\mathbf{v}_0-\mathbf{v}_b)\cdot\boldsymbol{\eta},\nabla\cdot\mathbf{v}_0\rangle_{\partial T} \\ &= (\nabla_w\cdot\mathbf{v}_h,\nabla\cdot\mathbf{v}_0)_T + \big\langle\mathbf{Q}_b^{k-1}\mathbf{v}_0-\mathbf{v}_b,(\nabla\cdot\mathbf{v}_0)\boldsymbol{\eta}\big\rangle_{\partial T}. \end{aligned}$$

Then, trace inequality (2.4) and inverse estimate (2.5) yields

$$\|\nabla\cdot\mathbf{v}_0\|_T^2 \le C\left(\|\nabla_w\cdot\mathbf{v}_h\|_T\|\nabla\cdot\mathbf{v}_0\|_T + h_T^{-1/2}\|\mathbf{Q}_b^{k-1}\mathbf{v}_0-\mathbf{v}_b\|_{\partial T}\|\nabla\cdot\mathbf{v}_0\|_T\right),$$

which leads to Lemma 3.1.

Lemma 3.2. For any $\mathbf{v}_h=\{\mathbf{v}_0,\mathbf{v}_b\}\in\mathbf{V}_h$, we have

$$\left\|\mathbf{v}_0-\mathbf{Q}_b^{k-1}\mathbf{v}_0\right\|^2 \le Ch\|\mathbf{v}_0\|_{1,T}^2,\ T\in\mathcal{T}_h. \tag{3.6}$$

Proof. For any $\mathbf{v}_h=\{\mathbf{v}_0,\mathbf{v}_b\}\in\mathbf{V}_h$, we have

$$\begin{aligned}\|\mathbf{v}_0-\mathbf{Q}_b^{k-1}\mathbf{v}_0\| &\le \|\mathbf{v}_0-\mathbf{Q}_0^{k-1}\mathbf{v}_0\|_{\partial T}+\|\mathbf{Q}_0^{k-1}\mathbf{v}_0-\mathbf{Q}_b^{k-1}\mathbf{v}_0\|_{\partial T}\\ &\le C\big(\|\mathbf{v}_0-\mathbf{Q}_0^{k-1}\mathbf{v}_0\|_{\partial T}+\|\mathbf{Q}_0^{k-1}\mathbf{v}_0-\mathbf{v}_0\|_{\partial T}\big)\\ &\le C\|\mathbf{Q}_0^{k-1}\mathbf{v}_0-\mathbf{v}_0\|_{\partial T}.\end{aligned}$$

Now, by using the trace inequality (2.4) and Lemma 2.1, we obtain

$$\begin{aligned}\|\mathbf{Q}_0^{k-1}\mathbf{v}_0-\mathbf{v}_0\|_{\partial T}^2 &\le C\left(h_T^{-1}\|\mathbf{Q}_0^{k-1}\mathbf{v}_0-\mathbf{v}_0\|_T^2+h_T\|\nabla\big(\mathbf{Q}_0^{k-1}\mathbf{v}_0-\mathbf{v}_0\big)\|_T^2\right)\\ &\le Ch_T\|\mathbf{v}_0\|_{1,T}^2 \le Ch_T\big(\|\mathbf{v}_0\|_T^2+\|\nabla\cdot\mathbf{v}_0\|_T^2\big).\end{aligned}$$

This completes the rest of the proof.

Now, we turn our analysis towards the error estimates. For this purpose, we write

$$\mathbf{u}-\mathbf{u}_h=(\mathbf{u}-\mathbf{Q}_h\mathbf{u})+(\mathbf{Q}_h\mathbf{u}-\mathbf{u}_h),$$

where $\mathbf{Q}_h\mathbf{u}=\{\mathbf{Q}_0^k\mathbf{u},\mathbf{Q}_b^{k-1}\mathbf{u}\}$. For simplicity, we introduce following notation

$$\mathbf{e}_h:=(\mathbf{Q}_h\mathbf{u}-\mathbf{u}_h)=\{\mathbf{Q}_0^k\mathbf{u}-\mathbf{u}_0,\mathbf{Q}_b^{k-1}\mathbf{u}-\mathbf{u}_b\}. \tag{3.7}$$

Then, we derive following crucial error equation for $\mathbf{e}_h$.

Lemma 3.3. Let $\mathbf{e}_h$ be the error function given by (3.7). Then, for each $\mathbf{v}_h = \{\mathbf{v}_0, \mathbf{v}_b\} \in \mathbf{V}_h^0$, we obtain

$$\mathcal{A}_1(\mathbf{e}_h, \mathbf{v}_h) = R_1(\mathbf{u}, \mathbf{v}_h) + R_2(\mathbf{u}, \mathbf{v}_h) + \mathcal{S}(\mathbf{Q}_h\mathbf{u}, \mathbf{v}_h), \tag{3.8}$$

where bilinear maps $R_1(\cdot,\cdot)$ and $R_2(\cdot,\cdot)$ are given by

$$R_1(\mathbf{u}, \mathbf{v}_h) = \sum_{T\in\mathcal{T}_h} \left\langle \left(\alpha\nabla\cdot\mathbf{u} - \mathbb{Q}_h^{k-1}(\alpha\nabla\cdot\mathbf{u})\right)\boldsymbol{\eta}, \mathbf{v}_0 - \mathbf{v}_b \right\rangle_{\partial T}, \tag{3.9}$$

$$R_2(\mathbf{u}, \mathbf{v}_h) = \sum_{T\in\mathcal{T}_h} \left(\alpha\mathbb{Q}_h^{k-1}(\nabla\cdot\mathbf{u}) - \mathbb{Q}_h^{k-1}(\alpha\nabla\cdot\mathbf{u}), \nabla_w\cdot\mathbf{v}_h\right)_T. \tag{3.10}$$

Proof. For $\mathbf{v}_h = \{\mathbf{v}_0, \mathbf{v}_b\} \in \mathbf{V}_h^0$, testing equation (1.1) by $\mathbf{v}_0$, we arrive at

$$-\sum_{T\in\mathcal{T}_h} (\nabla(\alpha\nabla\cdot\mathbf{u}), \mathbf{v}_0)_T + \left(\mathbf{Q}_0^k(\beta\mathbf{u}), \mathbf{v}_0\right) = (\mathbf{f}, \mathbf{v}_0) \tag{3.11}$$

Then, apply integration by parts to obtain

$$\begin{aligned}(\mathbf{f}, \mathbf{v}_0) = \; & \left(\beta\mathbf{Q}_0^k\mathbf{u}, \mathbf{v}_0\right) + \textstyle\sum_{T\in\mathcal{T}_h} (\alpha\nabla\cdot\mathbf{u}, \nabla\cdot\mathbf{v}_0)_T \\ & -\textstyle\sum_{T\in\mathcal{T}_h} \langle \mathbf{v}_0 - \mathbf{v}_b, (\alpha\nabla\cdot\mathbf{u})\boldsymbol{\eta}\rangle_{\partial T}.\end{aligned} \tag{3.12}$$

Here, we have assumed that β is piecewise constant and the fact that

$$\sum_{T\in\mathcal{T}_h} \langle(\alpha\nabla\cdot\mathbf{u})\boldsymbol{\eta}, \mathbf{v}_b\rangle_{\partial T} = 0$$

Now, using the definition (2.2) for $\mathbf{v}_h$ with $l = k - 1$, we obtain

$$
\begin{aligned}
&\left(\mathbb{Q}_h^{k-1}(\alpha\nabla\cdot\mathbf{u}),\nabla_w\cdot\mathbf{v}_h\right)_T\\
&=-\left(\mathbf{v}_0,\nabla\left(\mathbb{Q}_h^{k-1}(\alpha\nabla\cdot\mathbf{u})\right)\right)_T+\left\langle\mathbf{v}_b,\mathbb{Q}_h^{k-1}(\alpha\nabla\cdot\mathbf{u})\boldsymbol{\eta}\right\rangle_{\partial T}\\
&=\left(\nabla\cdot\mathbf{v}_0,\mathbb{Q}_h^{k-1}(\alpha\nabla\cdot\mathbf{u})\right)_T-\left\langle\left(\mathbb{Q}_h^{k-1}(\alpha\nabla\cdot\mathbf{u})\right)\boldsymbol{\eta},\mathbf{v}_0\right\rangle_{\partial T}\\
&+\left\langle\mathbb{Q}_h^{k-1}(\alpha\nabla\cdot\mathbf{u})\boldsymbol{\eta},\mathbf{v}_b\right\rangle_{\partial T}\\
&=\left(\nabla\cdot\mathbf{v}_0,\mathbb{Q}_h^{k-1}(\alpha\nabla\cdot\mathbf{u})\right)_T-\left\langle\left(\mathbb{Q}_h^{k-1}(\alpha\nabla\cdot\mathbf{u})\right)\boldsymbol{\eta},\mathbf{v}_0-\mathbf{v}_b\right\rangle_{\partial T}\\
&=(\nabla\cdot\mathbf{v}_0,\alpha\nabla\cdot\mathbf{u})_T-\left\langle\left(\mathbb{Q}_h^{k-1}(\alpha\nabla\cdot\mathbf{u})\right)\boldsymbol{\eta},\mathbf{v}_0-\mathbf{v}_b\right\rangle_{\partial T}. \qquad (3.13)
\end{aligned}
$$

Then, combine (3.13) and (3.12) to obtain

$$
\begin{aligned}
(\mathbf{f},\mathbf{v}_0)=\ &\left(\beta\mathbf{Q}_0^k\mathbf{u},\mathbf{v}_0\right)+\left(\mathbb{Q}_h^{k-1}(\alpha\nabla\cdot\mathbf{u}),\nabla_w\cdot\mathbf{v}_h\right)\\
&-\textstyle\sum_{T\in\mathcal{T}_h}\left\langle\left(\alpha\nabla\cdot\mathbf{u}-\mathbb{Q}_h^{k-1}(\alpha\nabla\cdot\mathbf{u})\right)\boldsymbol{\eta},\mathbf{v}_0-\mathbf{v}_b\right\rangle_{\partial T}\\
=\ &\left(\beta\mathbf{Q}_0^k\mathbf{u},\mathbf{v}_0\right)+\left(\alpha\mathbb{Q}_h^{k-1}(\nabla\cdot\mathbf{u}),\nabla_w\cdot\mathbf{v}_h\right)\\
&-\left(\alpha\mathbb{Q}_h^{k-1}(\nabla\cdot\mathbf{u})-\mathbb{Q}_h^{k-1}(\alpha\nabla\cdot\mathbf{u}),\nabla_w\cdot\mathbf{v}_h\right)\\
&-\textstyle\sum_{T\in\mathcal{T}_h}\left\langle\left(\alpha\nabla\cdot\mathbf{u}-\mathbb{Q}_h^{k-1}(\alpha\nabla\cdot\mathbf{u})\right)\boldsymbol{\eta},\mathbf{v}_0-\mathbf{v}_b\right\rangle_{\partial T}\\
=\ &\left(\beta\mathbf{Q}_0^k\mathbf{u},\mathbf{v}_0\right)+(\alpha\nabla_w\cdot(\mathbf{Q}_h\mathbf{u}),\nabla_w\cdot\mathbf{v}_h)\\
&-R_1(\mathbf{u},\mathbf{v}_h)-R_2(\mathbf{u},\mathbf{v}_h). \qquad (3.14)
\end{aligned}
$$

In the last equality, we have used identity (2.3).

Next, by adding $\mathcal{S}(\mathbf{Q}_h\mathbf{u},\mathbf{v}_h)$ both sides to (3.14), we obtain

$$\mathcal{A}_1(\mathbf{Q}_h\mathbf{u},\mathbf{v}_h)=(\mathbf{f},\mathbf{v}_0)+R_1(\mathbf{u},\mathbf{v}_h)+R_2(\mathbf{u},\mathbf{v}_h)+\mathcal{S}(\mathbf{Q}_h\mathbf{u},\mathbf{v}_h). \qquad (3.15)$$

Finally, subtracting (3.4) from (3.15) leads to desire result.

Next result deals with the bounds for the terms in error equation (3.8).

Lemma 3.4. Let $\mathbf{u}\in[H^{k+1}(\Omega)]^2$. Then, for any $\mathbf{v}_h=\{\mathbf{v}_0,\mathbf{v}_b\}\in\mathbf{V}_h^0$, we obtain

$$|R_1(\mathbf{u},\mathbf{v}_h)| \leq C(\| \alpha \|_{k,\infty})h^k \| \mathbf{u} \|_{k+1} \|\mathbf{v}_h\|_1, \tag{3.16}$$
$$|R_2(\mathbf{u},\mathbf{v}_h)| \leq C(\| \alpha \|_{1,\infty})h^{k+1} \| \mathbf{u} \|_{k+1} \|\mathbf{v}_h\|_1, \tag{3.17}$$
$$|\mathcal{S}(\mathbf{Q}_h\mathbf{u},\mathbf{v}_h)| \leq Ch^k \| \mathbf{u} \|_{k+1} \|\mathbf{v}_h\|_1. \tag{3.18}$$

Proof. For the estimate (3.16), we first note that

$$\begin{aligned} R_1(\mathbf{u},\mathbf{v}_h) = & \sum_{T\in\mathcal{T}_h} \langle (\alpha(\nabla\cdot\mathbf{u}) - \mathbb{Q}_h^{k-1}(\alpha\nabla\cdot\mathbf{u}))\boldsymbol{\eta}, \mathbf{v}_0 - \mathbf{v}_b\rangle_{\partial T} \\ = & \sum_{T\in\mathcal{T}_h} \langle (\alpha(\nabla\cdot\mathbf{u}) - \mathbb{Q}_h^{k-1}(\alpha\nabla\cdot\mathbf{u}))\boldsymbol{\eta}, \mathbf{v}_0 - \mathbf{Q}_b^{k-1}\mathbf{v}_0\rangle_{\partial T} \\ & + \sum_{T\in\mathcal{T}_h} \langle (\alpha(\nabla\cdot\mathbf{u}) - \mathbb{Q}_h^{k-1}(\alpha\nabla\cdot\mathbf{u}))\boldsymbol{\eta}, \mathbf{Q}_b^{k-1}\mathbf{v}_0 - \mathbf{v}_b\rangle_{\partial T} \\ := & R_{1,1} + R_{1,2}. \end{aligned} \tag{3.19}$$

For $R_{1,1}$, we use the Cauchy-Schwarz inequality, Lemmas 3.1-3.2 and then trace inequality (2.4) along with Lemma 2.1 to obtain

$$\begin{aligned} |R_{1,1}| = & \left|\sum_{T\in\mathcal{T}_h} \langle (\alpha(\nabla\cdot\mathbf{u}) - \mathbb{Q}_h^{k-1}(\alpha\nabla\cdot\mathbf{u}))\boldsymbol{\eta}, \mathbf{v}_0 - \mathbf{Q}_b^{k-1}\mathbf{v}_0\rangle_{\partial T}\right| \\ \leq & C\sum_{T\in\mathcal{T}_h} h_T^{\frac{1}{2}}\|\alpha(\nabla\cdot\mathbf{u}) - \mathbb{Q}_h^{k-1}(\alpha\nabla\cdot\mathbf{u})\|_{\partial T}(\|\mathbf{v}_0\|_T + \|\nabla\cdot\mathbf{v}_0\|_T) \\ \leq & C\left(\sum_{T\in\mathcal{T}_h} h_T\|\alpha(\nabla\cdot\mathbf{u}) - \mathbb{Q}_h^{k-1}(\alpha\nabla\cdot\mathbf{u})\|_{\partial T}^2\right)^{\frac{1}{2}} \\ & \times\left(\sum_{T\in\mathcal{T}_h} (\|\mathbf{v}_0\|_T^2 + \|\nabla\cdot\mathbf{v}_0\|_T^2)\right)^{\frac{1}{2}} \\ \leq & C(\| \alpha \|_{k,\infty})h^k \| \mathbf{u} \|_{k+1} \|\mathbf{v}_h\|_1. \end{aligned} \tag{3.20}$$

Similar arguments yield

$$
\begin{aligned}
|R_{1,2}| = & \left| \sum_{T\in\mathcal{T}_h} \langle (\alpha(\nabla\cdot\mathbf{u}) - \mathbb{Q}_h^{k-1}(\alpha\nabla\cdot\mathbf{u}))\boldsymbol{\eta}, \mathbf{Q}_b^{k-1}\mathbf{v}_0 - \mathbf{v}_b\rangle_{\partial T} \right| \\
\leq & \ C \sum_{T\in\mathcal{T}_h} \|\alpha(\nabla\cdot\mathbf{u}) - \mathbb{Q}_h^{k-1}(\alpha\nabla\cdot\mathbf{u})\|_{\partial T} \|\mathbf{Q}_b^{k-1}\mathbf{v}_0 - \mathbf{v}_b\|_{\partial T} \\
\leq & \ C \left(\sum_{T\in\mathcal{T}_h} h_T \|\alpha(\nabla\cdot\mathbf{u}) - \mathbb{Q}_h^{k-1}(\alpha\nabla\cdot\mathbf{u})\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
& \times \left(\sum_{T\in\mathcal{T}_h} h_T^{-1} \|\mathbf{Q}_b^{k-1}\mathbf{v}_0 - \mathbf{v}_b\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
\leq & \ C(\|\alpha\|_{k,\infty}) h^k \|\mathbf{u}\|_{k+1} \|\mathbf{v}_h\|_1. \qquad (3.21)
\end{aligned}
$$

Then, estimate (3.16) follows from (3.19)-(3.21).

It follows from Lemma 2.1 that

$$
\begin{aligned}
|R_2(\mathbf{u},\mathbf{v}_h)| &= |(\alpha\mathbb{Q}_h^{k-1}(\nabla\cdot\mathbf{u}) - \alpha\nabla\cdot\mathbf{u}, \nabla_w\cdot\mathbf{v}_h)| \\
&\leq |(\mathbb{Q}_h^{k-1}(\nabla\cdot\mathbf{u}) - \nabla\cdot\mathbf{u}, (\alpha-\bar{\alpha})\nabla_w\cdot\mathbf{v}_h)| \\
&\leq C(\|\alpha\|_{1,\infty}) h |(\mathbb{Q}_h^{k-1}(\nabla\cdot\mathbf{u}) - \nabla\cdot\mathbf{u}, \nabla_w\cdot\mathbf{v}_h)| \\
&\leq C(\|\alpha\|_{1,\infty}) h^{k+1} \|\mathbf{u}\|_{k+1} \|\mathbf{v}_h\|_1. \qquad (3.22)
\end{aligned}
$$

Here, $\bar{\alpha}$ denotes the average of α and on each element $T\in\mathcal{T}_h$, following inequality holds true (see, [14]).

$$
\|\bar{\alpha}-\alpha\|_{L^\infty(T)} \leq Ch \|\nabla\alpha\|_{L^\infty(T)}. \qquad (3.23)
$$

To estimate the stabilizer term, we use (3.2), inequality (2.4) and Lemma 2.1 to arrive at

$$
\begin{aligned}
|\mathcal{S}(\mathbf{Q}_h\mathbf{u},\mathbf{v}_h)| &= \left|\sum_{T\in\mathcal{T}_h} h_T^{-1}\langle \mathbf{Q}_b^{k-1}(\mathbf{Q}_0^k\mathbf{u}) - \mathbf{Q}_b^{k-1}\mathbf{u}, \mathbf{Q}_b^{k-1}\mathbf{v}_0 - \mathbf{v}_b\rangle_{\partial T}\right| \\
&= \left|\sum_{T\in\mathcal{T}_h} h_T^{-1}\langle \mathbf{Q}_b^{k-1}(\mathbf{Q}_0^k\mathbf{u} - \mathbf{u}), \mathbf{Q}_b^{k-1}\mathbf{v}_0 - \mathbf{v}_b\rangle_{\partial T}\right| \\
&= \left|\sum_{T\in\mathcal{T}_h} h_T^{-1}\langle \mathbf{Q}_0^k\mathbf{u} - \mathbf{u}, \mathbf{Q}_b^{k-1}\mathbf{v}_0 - \mathbf{v}_b\rangle_{\partial T}\right| \\
&\le \sum_{T\in\mathcal{T}_h} h_T^{-1}\|\mathbf{Q}_0^k\mathbf{u} - \mathbf{u}\|_{\partial T}\|\mathbf{Q}_b^{k-1}\mathbf{v}_0 - \mathbf{v}_b\|_{\partial T} \\
&\le \left(\sum_{T\in T_h} \left(h_T^{-2}\|\mathbf{Q}_0^k\mathbf{u} - \mathbf{u}\|_T^2 + \|\nabla(\mathbf{Q}_0^k\mathbf{u} - \mathbf{u})\|_T^2\right)\right)^{\frac{1}{2}} \\
&\quad \times\left(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|\mathbf{Q}_b^{k-1}\mathbf{v}_0 - \mathbf{v}_b\|_{\partial T}^2\right)^{\frac{1}{2}} \\
&\le Ch^k \parallel \mathbf{u} \parallel_{k+1} \|\mathbf{v}_h\|_1.
\end{aligned}
$$

This completes the proof of Lemma 3.4.

Theorem 3.1. Let $\mathbf{u} \in [H^{k+1}(\Omega)]^2$ be the exact solution for (1.1)-(1.2) and $\mathbf{u}_h \in \mathbf{V}_h^0$ be the weak Galerkin finite element solution of (3.4). Then, there exists a constant C such that

$$\|\mathbf{Q}_h\mathbf{u} - \mathbf{u}_h\|_1 \le Ch^k \parallel \mathbf{u} \parallel_{k+1}. \tag{3.24}$$

Proof. By letting $\mathbf{v}_h = \mathbf{e}_h$ in the error equation (3.8), we have

$$\|\mathbf{e}_h\|_1^2 \le |R_1(\mathbf{u},\mathbf{e}_h)| + |R_2(\mathbf{u},\mathbf{e}_h)| + |\mathcal{S}(\mathbf{Q}_h\mathbf{u},\mathbf{e}_h)|. \tag{3.25}$$

Here, $\mathbf{e}_h$ is as defined in (3.7). Then the desired estimate (3.24) follows immediately from Lemma 3.4 and above estimate (3.25).

4. Numerical Section

In this section, we have tested various numerical examples for the $\mathbf{H}(\text{div};\Omega)$ elliptic problem (1.1)-(1.2) in Ω, where $\Omega \subset \mathbb{R}^2$. All computations are carried out using the MATLAB software.

For a given finite number of successive iterations (indexed by i), let e_i be the error corresponding to suitable norm on the i-th iteration, and h_i is corresponding mesh size. Then expected order of convergence (EOC) can be defined by

$$\text{EOC}\,(e_i) = \frac{\log\left(\frac{e_{i+1}}{e_i}\right)}{\log\left(\frac{h_{i+1}}{h_i}\right)},$$

and the order of the convergence is defined as $\lim_{i\to\infty} \text{EOC}\,(e_i)$.

Example 4.1. Convergence test for $\mathbf{H}(\text{div};\Omega)$-elliptic problem on mixed mesh: Consider the problem (1.1)-(1.2) in $\Omega = [0,1] \times [0,1]$. The exact solution is set as

$$\mathbf{u} = (u_1, u_2) = (\sin(x+y) + \cos(x+y), \exp(x)\cos(\pi y)).$$

The right-hand side $\mathbf{f}$ can be evaluated from the exact solution $\mathbf{u}$ and the coefficients $\alpha = x^2 + y^2 + 1$ and $\beta = xy + 3$. Mixed meshes are used in this example,

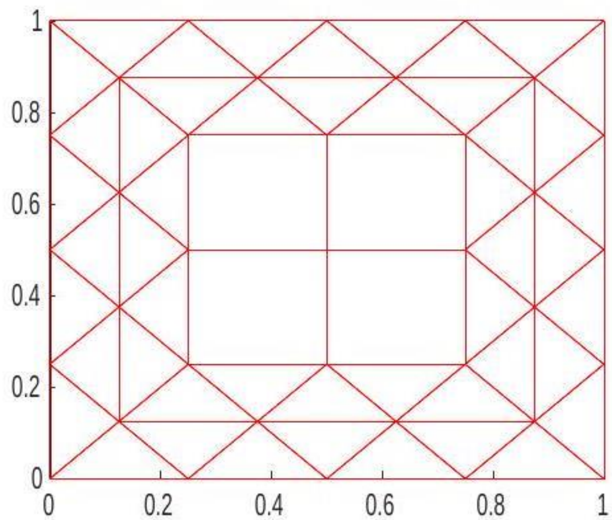

Figure 4.1: Initial mixed mesh.

which is depicted in Figure 4.1. The errors, for the proposed weak Galerkin algorithm, with respect to discrete H^1 norm are reported in Table 4.1. WG solutions are shown in Figures 4.2-4.3 for $k = 2$ and $k=3$ with $h = 1/64$.

Table 4.1: Errors and convergence profile for WG solution in Example 4.1.

k	h	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	Order
1	$\Vert \mathbf{e}_h \Vert_1$	2.4616e + 00	1.3531e + 00	7.0642e − 01	3.5875e − 01	1.8027e − 01	1
2	$\Vert \mathbf{e}_h \Vert_1$	6.6600e − 01	1.7546e − 01	4.4610e − 02	1.1220e − 02	2.8120e − 03	2
3	$\Vert \mathbf{e}_h \Vert_1$	1.1062e − 01	1.4333e − 02	1.8097e − 03	2.2689e − 04	2.8390e − 05	3

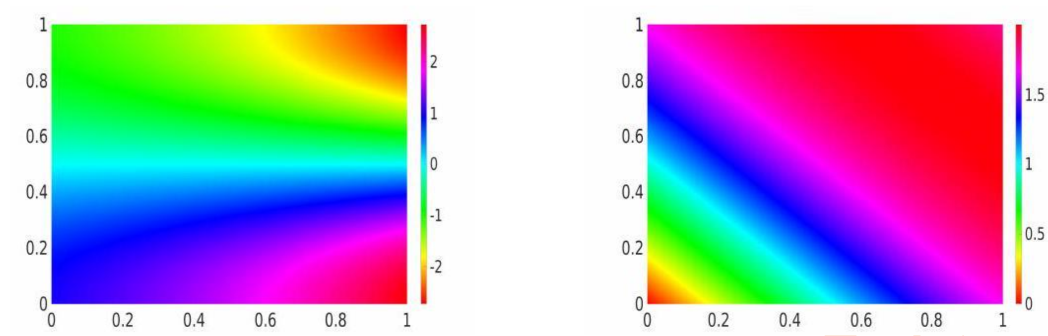

Figure 4.2: (Test Example 4.1, *k=2*) Component-wise surface plots of WG solution $\mathbf{u}_h$ for WGALG-1. Plot for the first-component of $\mathbf{u}_h$ (left) and plot for second component of $\mathbf{u}_h$ (right).

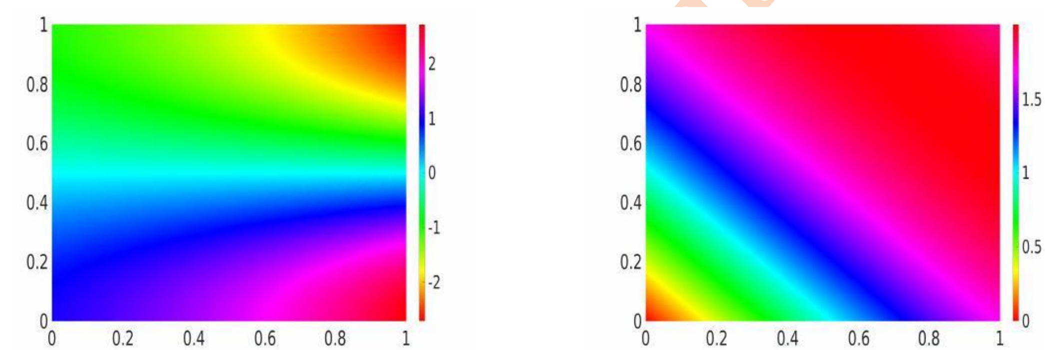

Figure 4.3: (Test Example 4.1, *k=3*) Component-wise surface plots of WG solution $\mathbf{u}_h$ for WGALG-1. Plot for the first-component of $\mathbf{u}_h$ (left) and plot for second component of $\mathbf{u}_h$ (right).

4. Conclusion

In this paper, we have developed a weak Galerkin scheme for solving the grad-div elliptic problems. Convergence analysis has been carried out on the local weak Galerkin space $(\mathcal{P}_k^2, \mathcal{P}_{k-1}^2, \mathcal{P}_{k-1})$. Optimal order of convergence is established for the WG approximation in H^1-like norm.

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