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ON THE REFLECTION OF WATER WAVES BY A CURVED WALL IN PRESENCE OF SURFACE TENSION

PARTHA AGASTI
Department of Mathematics, SBSS Mahavidyalaya, Goaltore-721128, India
Email: parthaagasti15@gmail.com

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Abstract

Present analysis is concerned with the problem of water waves progressing obliquely towards a rigid curved wall, in deep water. Considering the effect of surface tension at the free surface, the problem under consideration is attacked for solution by a standard Perturbation technique along with the application of Havelock's expansion [1] of water wave potential. The first order corrections to the velocity potential and reflection coefficient are obtained in terms of integral involving the shape function. These corrections are explicitly calculated by assuming two particular shapes of the curved wall.

Keywords: linear theory, curved wall, velocity potential, perturbation theory, irrotational flow**.**

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1. Introduction

The purpose of the present investigation is to find an analytical solution for the water wave potential of obliquely incident water waves progressing towards a curved wall, in deep water in presence of surface tension. As there is no mechanism to absorb the incoming energy in the inviscid fluid system, so total reflection of waves by the wall is assumed. The interaction of surface waves involving a vertical wall and few of its' generalizations in the presence of surface tension have attracted the attention of many scientists and studied extensively (cf. [2]-[6]). However, the problems involving a curved wall have not received much care. The first problem in this field was considered by Shaw (cf.[7]) where he employed a technique based on perturbation theory that involves the solution of a singular integral equation to obtain the first order corrections to the reflection and transmission co-efficients associated with a surface piercing nearly vertical barrier in deep water. Mandal and Kar (cf.[8]) studied the problem of reflection of water waves by a nearly vertical wall and they used simplified perturbation technique. Since then, attempts have been made to study this class of water wave problems by applying different mathematical methods (cf.[9]-[12]).

Reflection of wave from beaches has an immense importance for understanding the near shore zone and for improving coastal structure design. The level of energy flux dissipation which occurs on a beach depends on the magnitude of the wave reflection from the beach. Thus, in an ancillary approach, wave reflection influences many coastal processes such as run-up which, in turn, determines coastal design criteria such as the height of a sea wall or flood protection dune. A curved wall is perhaps the simplest model of this type of beach.

If the wall is perfectly vertical, its effect on the source is equivalent to another source situated at the image point of the original source with respect to the vertical wall. However, due to the curved nature of the wall, there will be additional contributions. These contributions have been found, in this study, upto the first order term to the reflection coefficient and velocity potential by using a simple perturbation analysis followed by an appropriate Havelock's expansion of water wave potential for deep water including surface tension effect. Considering two particular shapes of the curved wall, viz. $f(y) = y\exp(-\lambda y)$ and $f(y) = a\sin\lambda y$, these corrections are also calculated.

2. Mathematical Formulation of the Problem

Consider the three-dimensional irrotational motion of an inviscid, incompressible liquid of density, under the action of gravity g only. A rectangular cartesian co-ordinate system is chosen so that the y-axis is taken vertically downwards into the liquid and assume that the liquid is bounded on the left by the curved wall $x = \varepsilon f(y), 0 < y < \infty$, where ε is a small dimensionless positive quantity and $f(y)$ is bounded and continuous in $0 < y < \infty$ with $f(0) = 0$ so that $y = 0, x > 0$ is the undisturbed free surface. The origin is taken at a point on the line of intersection where the curved wall and the free surface meet. Usual assumption of the linear theory ensures the existence of a velocity potential $\Phi(x, y, z, t)$, which represents progressive waves moving towards the shore line (i.e., the z-axis) such that the wave crests at large distance from the shore tend to straight line which make an arbitrary angle θ with the z-axis.

Assuming the motion to be of simple harmonic in time with circular frequency σ and of small amplitude, the velocity potential of the liquid can be described by

$$\Phi(x, y, z, t) = Re[\phi(x, y)\exp\{-i(\sigma t + \omega z)\}]$$

where $\omega = \alpha_0 sin\theta$, α_0 is defined later.

Using linear theory, the time independent potential function $\phi(x, y)$ satisfies the following boundary value problem (BVP):

Two dimensional modified Helmholtz's equation:

$$(\nabla^2 - \omega^2)\,\phi = 0 \quad \text{in the flow domain,} \tag{2.1}$$

where ∇^2 is the two-dimensional Laplacian.

Free surface condition:

$$K\phi + \phi_y + M\phi_{yyy} = 0 \;\; on \quad y = 0, x > 0\,, \tag{2.2}$$

where $K = \frac{\sigma^2}{g}$ is the wave number and $M = \frac{\tau}{\rho g}$, τ is the coefficient of surface tension.

Rigid body condition:

$$\phi_n = 0 \quad \text{on } x = \varepsilon f(y), \qquad y > 0, \tag{2.3}$$

where n denotes the outward drawn normal to the surface of the curved wall.

Sea-bed condition:

$$\nabla\phi \to 0 \text{ as } y \to \infty. \tag{2.4}$$

In addition to the above conditions, $\phi(x, y)$ is also required to satisfy the requirement that

$$\phi \to \phi_{inc}(x, y) + R\phi_{inc}(-x, y) \quad as \quad x \to \infty \tag{2.5}$$

where a train of surface waves represented by $\phi_{inc}(x, y) = \exp(-\alpha_0 y - i\eta x)$ is incident from positive infinity on the curved wall, R is the reflection coefficient, $\eta = \alpha_0 \cos\theta$ and α_0 is the unique real root (cf.[9]) of the cubic equation $x(1 + Mx^2) - K = 0$.

Since we have assumed that the parameter ε is very small, thus neglecting $O(\varepsilon^2)$ terms, the boundary condition (2.3) on the curved wall can be expressed, in approximate form, on $x = 0$ as (cf.[10])

$$\phi_x(0,y) - \varepsilon \frac{d}{dy}\{f(y)\phi_y(0,y)\} = 0 \text{ for } y > 0. \qquad (2.6)$$

3. Solution of the Problem

The form of the boundary condition (2.6) suggests that we may assume the following perturbational expansion, in terms of the small parameter ε, for the unknown function $\phi(x,y)$ and R respectively as

$$\left.\begin{aligned}\phi(x,y,\varepsilon) &= \phi_0(x,y) + \varepsilon\phi_1(x,y) + O(\varepsilon^2)\\ R(\varepsilon) &= R_0 + \varepsilon R_1 + O(\varepsilon^2)\end{aligned}\right\}. \qquad (3.1)$$

Our intention is to evaluate ϕ_0, R_0 and ϕ_1, R_1 as we are interested in finding only upto the first order corrections to the velocity potential and reflection coefficient.

Substituting the expansion (3.1) into equations (2.1), (2.2), (2.4), (2.5) and (2.6) and equating coefficients of ε^0 and ε from both sides, we find that the functions $\phi_0(x,y)$ and $\phi_1(x,y)$ must be the solution of the following two independent BVPs:

BVP-I: The function $\phi_0(x,y)$ satisfying:

$$(\nabla^2 - \omega^2)\phi_0 = 0 \text{ in the flow domain,}$$

$$K\phi_0 + \phi_{0_y} + M\phi_{0_{yyy}} = 0 \text{ on } y = 0, x > 0,$$
$$\phi_{0_x} = 0 \text{ on } x = 0, y > 0,$$
$$\nabla\phi_0 \to 0 \text{ as } y \to \infty,$$

$$\phi_0 \to \phi_{inc}(x,y) + R_0\phi_{inc}(-x,y) \text{ as } x \to \infty.$$

Certainly $\phi_0 = \phi_{inc}(x,y) + \phi_{inc}(-x,y)$ (3.2)

so that $R_0 = 1.$

BVP-II: The function $\phi_1(x,y)$ satisfies:

$$(\nabla^2 - \omega^2)\phi_1 = 0 \text{ in the flow domain,}$$

$$K\phi_1 + \phi_{1_y} + M\phi_{1_{yyy}} = 0 \text{ on } y = 0, x > 0,$$

$$\phi_{1_x} = \frac{d}{dy}\{f(y)\phi_{0_y}\} \text{ on } x = 0, \quad y > 0, \qquad (3.3)$$

$$\nabla\phi_1 \to 0 \text{ as } y \to \infty,$$

$$\phi_1 \to R_1 \phi_{inc}(-x, y) \ \text{ as } \ x \to \infty.$$

ϕ_1 and R_1 are the first order corrections to the velocity potential and reflection coefficient respectively in BVP-II which are to be determined.

Assume that

$$\frac{d}{dy}\left\{f(y)\phi_{0_y}\right\} = g(y) \ \text{ on } \ x = 0, \qquad y > 0, \tag{3.4}$$

so that from (3.3) we have $\quad \phi_{1_x} = g(y) \ \text{ on } \ x = 0, \ y > 0. \qquad (3.5)$

We employ the Havelock's [1] expansion of water wave potential to solve for $\phi_1(x, y)$.

Thus $\phi_1(x, y)$ can be expanded as

$$\phi_1(x, y) = R_1 \exp(-\alpha_0 y + i\eta x) \ + \int_o^\infty h(k)\{k(1 - Mk^2)\cos ky - K \sin ky\}$$

$$\times \ \exp\{-(k^2 + \omega^2)^{\frac{1}{2}} x\} dk, x > 0\,. \tag{3.6}$$

Using the condition (3.5) we have

$$g(y) = i\eta R_1 \exp(-\alpha_0 y) - \int_o^\infty (k^2 + \omega^2)^{\frac{1}{2}} h(k)\{k(1 - Mk^2)\cos ky - K \sin ky\}$$

$$\times \ \exp\{-(k^2 + \omega^2)^{\frac{1}{2}} x\} dk, \qquad y > 0.$$

Hence Havelock's inversion theorem gives

$$\frac{iR_1 \cos\theta}{2} = \int_0^\infty g(y) \exp(-\alpha_0 y) dy \tag{3.7}$$

and

$$h(k) = \frac{-2}{\pi\{k^2(1 - Mk^2)^2 + K^2\}(k^2 + \omega^2)^{\frac{1}{2}}}$$

$$\times \int_o^\infty g(y)\{k(1 - Mk^2)\cos ky - K \sin ky\}\, dy. \tag{3.8}$$

Thus $g(y)$ can be found via (3.4) if $f(y)$ is known and hence R_1 and $h(k)$ can be obtained from (3.7) and (3.8). Therefore the general expression for ϕ_1, the first order correction to the velocity potential can be determined when the effect of surface tension is taken into consideration.

4. Particular Shapes of the Curved Wall

Let us consider two particular shapes for the curved wall viz. (i) $f(y) = a\sin\lambda y$ and (ii) $f(y) = y\exp(-\lambda y)$, as considered by Chakrabarti [11].

CASE - I: $f(y) = a\sin\lambda y$.

In this case we obtain (see Appendix A):

$$g(y) = 2a\alpha_0(\alpha_0\sin\lambda y - \lambda\cos\lambda y)\exp(-\alpha_0 y) \tag{4.1}$$

so that

$$R_1 = \frac{4ia\lambda{\alpha_0}^2}{(4{\alpha_0}^2 + \lambda^2)\cos\theta} \tag{4.2}$$

and

$$h(k) = \frac{2ak\alpha_0}{\pi\{k^2(1 - Mk^2)^2 + K^2\}(k^2 + \omega^2)^{\frac{1}{2}}}\left[K\left\{\frac{\lambda + k}{(\lambda + k)^2 + {\alpha_0}^2} + \frac{\lambda - k}{(\lambda - k)^2 + {\alpha_0}^2}\right\}\right.$$
$$\left. + k\alpha_0(1 - Mk^2)\left\{\frac{1}{(\lambda - k)^2 + {\alpha_0}^2} - \frac{1}{(\lambda + k)^2 + {\alpha_0}^2}\right\}\right] \tag{4.3}$$

CASE - II: $f(y) = y\exp(-\lambda y)$.

In this case we find (see Appendix B)

$$g(y) = 2\alpha_0\{(\lambda + \alpha_0)y - 1\}\exp\{-(\lambda + \alpha_0)y\} \tag{4.4}$$

so that

$$R_1 = \frac{4i{\alpha_0}^2}{(\lambda + 2\alpha_0)^2\cos\theta} \tag{4.5}$$

and

$$h(k) = \frac{4k\alpha_0[2k^2(1 - Mk^2)(\lambda + \alpha_0) + K(\lambda + \alpha_0)^2 - Kk^2]}{\pi(k^2 + \omega^2)^{\frac{1}{2}}\{k^2(1 - Mk^2)^2 + K^2\}\{(\lambda + \alpha_0)^2 + k^2\}^2} \tag{4.6}$$

5. Conclusions

A straight forward perturbation technique along with the application of Havelock's expansion is employed to find the first order corrections to the reflection coefficient and velocity potential for the reflection of a three dimensional surface water wave train by a curved wall in the presence of surface tension at the free surface. Analytical expressions for these corrections R_1 and ϕ_1 are also calculated by assuming two particular shape of the curved wall. In the absence of surface tension effect, corresponding results can also be derived simply by putting $\tau = 0$. The major advantage of the analysis described here is that the corrections for the corresponding two dimensional problem can be recovered by the substitution of $\theta = 0$. The problem discussed in the present paper seems to have some applications in coastal design criteria and to derive the solution of the problem considered here, total reflection of waves by the curved wall is assumed since there is no mechanism to absorb (or dissipate) the incoming energy in the inviscid fluid. Thus the reflection of waves is a physically possible phenomenon in any non-dissipating system.

Appendix A

Noting (3.2), we find

$$\phi_0(0,y) = 2\, exp(-\alpha_0 y). \tag{A.1}$$

Thus for $f(y) = a \sin\lambda y$, we obtain, using $(A.1)$ into (3.4)

$$g(y) = 2a\frac{d}{dy}\left\{\sin\lambda y \frac{\partial}{\partial y} exp(-\alpha_0 y)\right\}$$

$$= 2a\alpha_0(\alpha_0 \sin\lambda y - \lambda \cos\lambda y)exp(-\alpha_0 y). \tag{A.2}$$

Applying $(A.2)$ in (3.7) we get

$$\frac{iR_1 \cos\theta}{2} = 2a\alpha_0 \int_0^{\infty} (\alpha_0 \sin\lambda y - \lambda \cos\lambda y)\, exp(-2\alpha_0 y) dy,$$

$$= -\frac{2a\lambda\alpha_0{}^2}{(4\alpha_0{}^2 + \lambda^2)}.$$

Hence (4.2) gives the simplified form of R_1.

To find $h(k)$, we have to calculate the integral

$$\int_o^\infty g(y)\{k(1-Mk^2)\cos ky - K\sin ky\}\,dy = J\,(say), \tag{A.3}$$

where $g(y)$ is given by $(A.2)$.

Applying $(A.2)$ into $(A.3)$, we find

$$J = 2a\alpha_0\{KJ_1 - k(1-Mk^2)J_2\}, \tag{A.4}$$

where

$$J_1 = \int_0^\infty \sin ky \frac{d}{dy}\{exp(-\alpha_0 y)\sin\lambda y\}dy$$

$$= -\frac{k}{2}\int_0^\infty exp(-\alpha_0 y)\{\sin(\lambda+k)y + \sin(\lambda-k)y\}\,dy$$

$$= -\frac{k}{2}\left\{\frac{\lambda+k}{\alpha_0{}^2+(\lambda+k)^2} + \frac{\lambda-k}{\alpha_0{}^2+(\lambda-k)^2}\right\} \tag{A.5}$$

and

$$J_2 = \int_0^\infty \cos ky \frac{d}{dy}\{exp(-\alpha_0 y)\sin\lambda y\}dy$$

$$= \frac{k}{2}\int_0^\infty exp(-\alpha_0 y)\{\cos(\lambda-k)y - \cos(\lambda+k)y\}\,dy$$

$$= \frac{k}{2}\left\{\frac{\alpha_0}{\alpha_0{}^2+(\lambda-k)^2} - \frac{\alpha_0}{\alpha_0{}^2+(\lambda+k)^2}\right\}. \tag{A.6}$$

Utilizing $(A.5)$ and $(A.6)$ into $(A.4)$, the integral given by $(A.3)$ can be determined, and hence from (3.8), $h(k)$ can be found which is given by (4.3).

Appendix B

Assuming $f(y) = y\exp(-\lambda y)$, and exploiting $(A1.1)$ into (3.4), we find

$$g(y) = 2\frac{d}{dy}\left\{y\exp(-\lambda y)\frac{\partial}{\partial y}exp(-\alpha_0 y)\right\}$$

$$= 2\alpha_0\{(\lambda+\alpha_0)y-1\}\exp\{-(\lambda+\alpha_0)y\} \qquad (B.1)$$

Using $(B.1)$ into (3.7), we get

$$\frac{iR_1\cos\theta}{2} = 2\alpha_0\int_0^\infty \{(\lambda+\alpha_0)y-1\}\exp\{-(\lambda+2\alpha_0)y\}\mathrm{dy} = -\frac{2\alpha_0{}^2}{(\lambda+2\alpha_0)^2}$$

Therefore R_1 can be determined and is given by (4.5).

To evaluate $h(k)$, we have to calculate the integral given by $(A.3)$ where $g(y)$ is given in $(B.1)$.

Exploiting $(B.1)$ into $(A.3)$, we obtain

$$J = -2\alpha_0(J_3 - J_4 - J_5 + J_6), \qquad (B.2)$$

where

$$J_3 = k(1-Mk^2)\int_0^\infty \cos ky\exp\{-(\lambda+\alpha_0)y\}\mathrm{dy}$$

$$= \frac{k(1-Mk^2)(\lambda+\alpha_0)}{(\lambda+\alpha_0)^2+k^2} \qquad (\mathrm{B}.3)$$

$$J_4 = k(1-Mk^2)(\lambda+\alpha_0)\int_0^\infty y\cos ky\exp\{-(\lambda+\alpha_0)y\}\mathrm{dy}$$

$$= \frac{k(1-Mk^2)(\lambda+\alpha_0)\{(\lambda+\alpha_0)^2-k^2\}}{\{(\lambda+\alpha_0)^2+k^2\}^2} \qquad (\mathrm{B}.4)$$

$$J_5 = K\int_0^\infty \sin ky\exp\{-(\lambda+\alpha_0)y\}\mathrm{dy} = \frac{Kk}{(\lambda+\alpha_0)^2+k^2} \qquad (\mathrm{B}.5)$$

$$J_6 = K(\lambda+\alpha_0)\int_0^\infty y\sin ky\exp\{-(\lambda+\alpha_0)y\}\mathrm{dy}$$

$$= \frac{2Kk(\lambda+\alpha_0)^2}{\{(\lambda+\alpha_0)^2+k^2\}^2} \qquad (\mathrm{B}.6)$$

Introducing $(\mathrm{B}.3)-(\mathrm{B}.6)$ into $(\mathrm{B}.2)$ we get the integral $(\mathrm{A}.3)$ and thus we finally obtain the expression for $h(k)$ given by (4.6).

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