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MODULAR COLORING AFTER FUSION IN SOME PLANAR GRAPHS

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Abstract

For a connected graph G, let c: V (G) →$\mathbb{Z}_k$ (k ≥ 2) be a vertex coloring of G. The color sum σ(v) of a vertex v of G is defined as the sum in $\mathbb{Z}_k$ of the colors of the vertices in N (v) that is $\sigma(v) = \sum_{u\in N(v)} c(u) \pmod k$. The coloring c is called a modular k-coloring of G if $\sigma(x) \neq \sigma(y)$ in $\mathbb{Z}_k$ for all pairs of adjacent vertices x, y ∈ G. The modular chromatic number or simply the mc-number of G is the minimum k for which G has a modular k coloring. A fusion graph is an ordinary graph with joining of two vertices. Here we investigating several problems on finding the mc(G) after fusion of two vertices on graphs and provide their characterization in terms of complexity.

Keywords: Modular coloring, Modular chromatic number, Fusion, Wheel Graph, Fan graph, Helm graph.

MSC AMS Classification 2020: 05C15

1. Introduction

We are encouraged by the modular colorings and the modular chromatic number of different graphs, where the chromatic number is defined as the color sum of all the neighboring vertices in $\mathbb{Z}_k$. At this point of view, to the curiosity for minimizing the modular chromatic number, determined to fusion in certain vertices in some graphs. For a vertex v of a graph G, let N (v) denote the neighborhood of v (the set of adjacent vertices to vertex v). For a graph G without isolated vertices, let c: V (G) → $\mathbb{Z}_k$ (k ≥ 2) be a vertex coloring of G where adjacent vertices may be colored the same. The color sum σ(v) of a vertex v of G is defined as the sum in $\mathbb{Z}_k$ of the colors of the vertices in N(v), that is σ(v) $=\sum_{u\epsilon N(v)} c(u)$**[4, 5, 7]**. The coloring c is called a modular sum k-coloring or simply a modular k-coloring of G, if σ(x) ≠ σ(y) in $\mathbb{Z}_k$ for all pairs x, y of adjacent vertices of G. A coloring c is called modular coloring if c is a modular k-coloring for some integer k ≥ 2. The modular chromatic number mc(G) is the minimum k for which G has a modular k-coloring. This concept was introduced by Okamoto, Salehi and Zhang **[1, 2, 3, 6]**. In order to distinguish the vertices of a connected graph and to differentiate the adjacent vertices of a graph with the minimum number of colors, the concept of modular coloring was put forward by Okamoto, Salehi and Zhang **[3]**.

For many problems, fusion graphs are a remarkable straight forward and natural model, but they have hardly been studied. Fusion on a vertex v of a graph has the effect of removing all edges incident with the vertex u and v and joining those edges to the new vertex x. Next, we investigating several problems on finding the mc(G) after fusion of graphs and provide their characterization in terms of complexity. In this paper we find the modular chromatic number of wheel graph, friendship graph, Helm graph, Fan and gear graph after fusion on certain vertices at different levels.

2. Basic Results

The following theorem is needed for the paper's result to be supported.

Theorem 2.1

For H1 (1, D) i.e., wheel graph, D ≥ 4, D is even, mc (H1 (1, D)) = 3.

For H1 (1, D) i.e., wheel graph, D ≥ 3, D is odd, mc (H1 (1, D)) = 4. **[7]**

For any n ≥ 2, mc(Fn) = 3. **[8]**

3. Fusion of a graph

Definition 3.1. A vertex fusion G_f of a graph G is obtained by taking two vertices u and v of G, by a single vertex x such that every edge which is incident with either u or v in G is incident with x in G. A graph H is the fusion of a graph G with respect to the vertices u and v of G if V (H) = V (G) + 1, and E(H) ≤ E(G)**[1]**. The fusion of G with respect to v is denoted G_f (v). The operation of creating G_f (v) is called fusion on two vertices u and v in G.

For a cycle related graph let ℓ_0 be the center, ℓ_1 be the vertices in the circle and ℓ_2 be the vertices outside the circle at a distance two from the center. Let u∈ ℓ_0 be the center, v_1, v_2, v_3, ….be the vertices in ℓ_1 and w_1, w_2, w_3, …. be the vertices in ℓ_2.The vertices in ℓ_1 and ℓ_2 are taken in the clockwise direction. In this article fusion of two adjacent vertices only taken into consideration.

Theorem 3.2.

The modular coloring of a graph obtained after fusion of two vertices in a wheel graph is

(i) $mc(W_f(n))=mc(W_{n-1})$ if u, v ∈ ℓ_1.

(ii) $mc(W_f(n))=mc(F(n-1))$ if u∈ ℓ_0 and v∈ ℓ_1.

Proof: There arise two cases for the fusion of two adjacent vertices in a wheel.

Fusion of a wheel is denoted by $W_f(n)$

Case (i) Fusion of two adjacent vertices in level ℓ_1 of a wheel having n vertices in the cycle. By fusing two vertices which are adjacent in ℓ_1 of W_n is reduced to a wheel having n-1 vertices. i.e., $W_f(n)$ is W_{n-1}.

∴ $mc(W_f(n))=mc(W_{n-1})$.

Case (ii) Fusion of two adjacent vertices, one is in level ℓ_1 and the other is in level ℓ_0 of a wheel having n vertices in the cycle.

By fusing two vertices such that u∈ ℓ_0 and $v_i \in \ell_1$, for any i result in a Fan having n-1 vertices at level ℓ_1. ∴ mc(W_f(n))=mc(F(n-1)).

Theorem 3.3.

The modular coloring of a graph obtained after fusion of two vertices in a Fan graph is

(i) mc(F_f(n))=mc(F(n-1)) if u, v ∈ ℓ_1& if u ∈ ℓ_0 ; $v_1 \in \ell_1$.

(ii) mc(F_f(4k))=3 if u∈ ℓ_0 and $v_i \neq 0$ where $2 \leq i \leq 4k-1$.

Proof: Let u∈ ℓ_0 be the center; v_1, v_2, v_3, …. v_{4k} be the vertices in ℓ_1. The vertices in ℓ_1 is taken in the clockwise direction.

Case (i.a). Fusion of two adjacent vertices in level ℓ_1 of a Fan having n vertices.

By fusing two vertices which are adjacent in ℓ_1 of F(n)is reduced to a Fan F(n-1) having n-1 vertices. i.e., F_f(n) is F(n-1). ∴ mc(F_f(n))=mc(F(n-1)).

Case (i.b). Fusion of two adjacent vertices, one is the extreme vertex at level ℓ_1 and the other is in level ℓ_0 of a Fan having n vertices at ℓ_1. By fusing two vertices such that u∈ ℓ_0 and $v_i \in \ell_1$, for any i result in a Fan having n-1 vertices at level ℓ_1.

∴ mc(F_f(n))=mc(F(n-1)).

Case (ii). Let u∈ ℓ_0 be the center; v_1, v_2, v_3, ….v_{4k}be the vertices in ℓ_1 .The vertices in ℓ_1 are taken in the clockwise direction.

Sub case (ii .a). Fusion in F_f(4k); k=2, 5, 8, …..

Let $v_5 \neq 0$ fused with u∈ ℓ_0 fused to form a vertex x. Then, the new graph is modified as F_f(4k) where 4k-1 vertices in ℓ_1.

Consider a modular coloring c(v): V(F_f(4k)) →$\mathbb{Z}_3$ defined by

$$c(v)=\begin{cases} 0 & \text{otherwise} \\ 2 & \text{for } v_{2+4t} \in \ell_1 \; for\; t = 0,1,2,3,\ldots \end{cases}$$

Then $\sigma(v) = \begin{cases} 1 & \text{if } u \in \ell_0 \\ 2 \text{ for } v_t \in \ell_1 & \text{where } t = 1,3,5 \ldots \\ 0 & \text{elsewhere} \end{cases}$ here $\sigma(x) \neq \sigma(y)\ \forall\ x, y$ of adjacent vertices in $F_f(4k)$ for k=2,5,8….∴ mc($F_f(4k)$)=3.

Hence the proof.

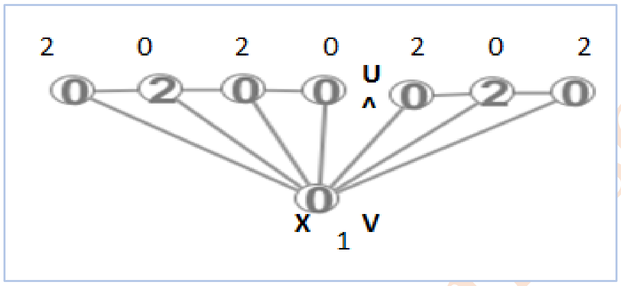

Figure 3.3.1 $F_f(8)$

Sub case (ii b). Fusion in $F_f(4k)$; k=3, 6, 9, …..

Let $v_5 \neq 0$ fused with u∈ ℓ_0 fused to form a vertex x. Then, the new graph is modified as $F_f(4k)$ where 4k-1 vertices in ℓ_1.

Consider a modular coloring c(v): V($F_f(4k)$) →$\mathbb{Z}_3$ defined by

$$c(v) = \begin{cases} 1 & \text{if } u \in \ell_0, v_{2+4t} \in \ell_1 \text{ or } t = 0,1,2 \ldots \\ 0 & \text{otherwise} \end{cases}$$

Then $\sigma(v) = \begin{cases} 0 & \text{if } u \in \ell_0 \\ 2 & \text{for } v_{2t} \text{ where } t = 1,2,3 \ldots \\ 1 & \text{elsewhere} \end{cases}$ here $\sigma(x) \neq \sigma(y)\ \forall\ x, y$ of adjacent vertices in $F_f(4k)$ for k=3,6,9….∴ mc($F_f(4k)$)=3.

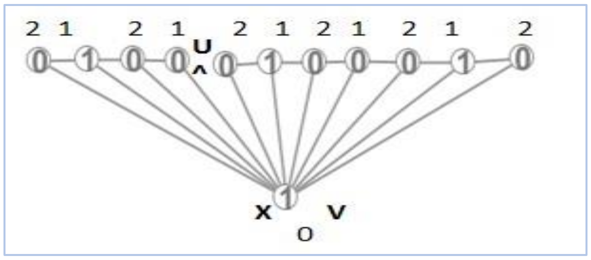

Figure 3.3.2 $F_f(12)$

Sub case (ii .c). Fusion in $F_f(4k)$;k=4,7,10,…..

Let $v_5 \neq 0$ fused with $u \in \ell_0$ fused to form a vertex x. Then, the new graph is modified as $F_f(4k)$ where 4k-1 vertices in ℓ_1.

Consider a modular coloring c(v): $V(F_f(4k)) \rightarrow \mathbb{Z}_3$ defined by

$$c(v)=\begin{cases} 1 & \text{if } u \in \ell_0 \\ 2 \text{ for } v_{2+4t} \in \ell_1 & \text{for } t = 0,1,2,3,\ldots \\ 0 & \text{otherwise} \end{cases}$$

Then $\sigma(v) = \begin{cases} 2 & \text{if } u \in \ell_0 \\ 1 \text{ for } v_{2t} \in \ell_1 & \text{where } t = 1,2,3 \ldots \\ 0 & \text{elsewhere} \end{cases}$ here $\sigma(x) \neq \sigma(y)\ \forall\ x, y$ of adjacent vertices in $F_f(4k)$ for k=4,7,10….

$$\therefore mc(F_f(4k))=3.$$

Hence the proof.

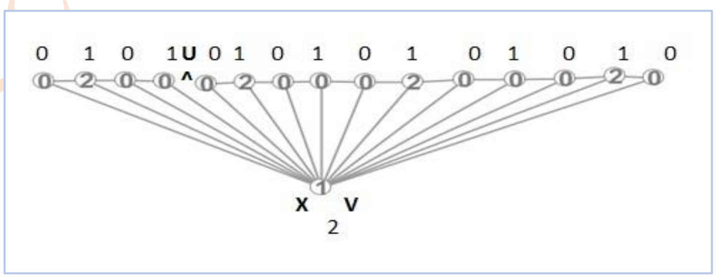

Figure 3.3.3 $F_f(16)$

Modular coloring after Fusion on Helm graph:

Let u∈ ℓ_0 be the center; v_1, v_2, v_3, ….v_n be the vertices in ℓ_1 and w_1, w_2, w_3, …. W_n be the vertices in ℓ_2. The vertices in ℓ_1 and ℓ_2 aretake**n** in the clockwise direction.

Fused Helm graph of n vertices is denoted by $H_f(n)$.

Fusion of two adjacent vertices in level ℓ=1 and level ℓ=2.

Theorem 3.4.

The graph obtained by fusing two vertices v_i and w_i where 1 ≤i≤ n ,then

(i) $mc(H_f(n))=3$ when n is even.

(ii) $mc(H_f(n))=4$ when n is odd.

Proof:

(i) Let $w_1 \in \ell_2$ and $v_1 \in \ell_1$ fused to become a vertex x in ℓ_1.
Consider a modular coloring c(v): $V(H_f(n)) \rightarrow \mathbb{Z}_3$ defined by

$$c(v)= \begin{cases} 1 & \text{if } u \in \ell_0, w_{2k} \in \ell_2 \ for\ k = 1,2,3 \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Then } \sigma(v) = \begin{cases} 1 & \text{for } v_{2k+1} \in \ell_1 \text{for k} = 0,1,2 \ldots \\ 2 & \text{for } v_{2k} \in \ell_1 \text{ where k} = 1,2,3 \ldots \\ 0 & \text{elsewhere} \end{cases}$$

here $\sigma(x) \neq \sigma(y)\ \forall\ x, y$ of adjacent vertices in $H_f(n)$

∴ $mc(H_f(n))=3$ for n is even. Hence the proof.

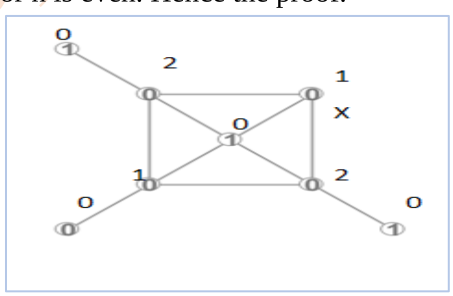

Figure 3.4.1 $H_f(4)$

(ii) Let $w_1 \in \ell_2$ and $v_1 \in \ell_1$ fused to become a vertex x in ℓ_1.

Consider a modular coloring $c(v):V(H_f(n)) \rightarrow \mathbb{Z}_4$ defined by

$$c(v)=\begin{cases} 1 & \text{if } u \in \ell_0, w_{2k+1} \in \ell_2 \text{ for } k = 1,2,3 \ldots \\ 2 & \text{for } w_2 \in \ell_2 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Then } \sigma(v) = \begin{cases} 1 & \text{for } v_1, v_{4+2k} \in \ell_1 \text{ for } k = 0,1,2 \ldots \\ 2 & \text{for } v_{2k+1} \in \ell_1 \text{ where } k = 1,2,3 \ldots \\ 3 & \text{for } v_2 \in \ell_1 \\ 0 & \text{elsewhere} \end{cases}$$

here $\sigma(x) \neq \sigma(y)\ \forall\ x, y$ of adjacent vertices in $H_f(n)$

$\therefore$ $mc(H_f(n))=4$ for n is odd. Hence the proof.

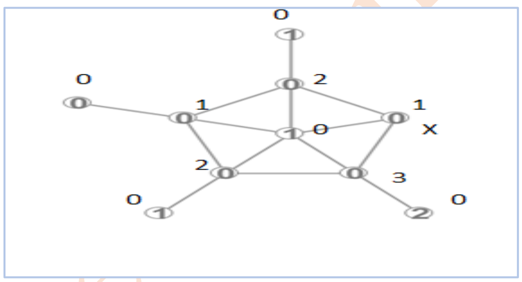

Figure 3.4.2$H_f(5)$

Fusion of two adjacent vertices in level ℓ=1.

Theorem 3.5.

The graph obtained by fusing two vertices v_1 and v_2, then

(i) $mc(H_f(2))=2$.

(ii) $mc(H_f(3+2t))=3$ for t =0,1,2,…

(iii) $mc(H_f(2+2t))=4$ for t = 0,1,2,3…

Proof:

(i) Let v_1 and v_2 fused to become a vertex x in ℓ_1.

Consider a modular coloring c(v): V(H_f(2)) →$\mathbb{Z}_2$ defined by $c(v)=\begin{cases}1 & \text{if } w_1 \in \ell_2\\ 0 & \text{otherwise}\end{cases}$

Then $\sigma(v)=\begin{cases}1 & \text{for } x \in \ell_1\\ 0 & \text{elsewhere}\end{cases}$ here $\sigma(x) \neq \sigma(y)\ \forall\ x, y$ of adjacent vertices in H_f(2).

∴ mc(H_f(2))=2 . Hence the proof.

(ii) Let v_1 and v_2 fused to become a vertex x in ℓ_1. Consider a modular coloring c(v): V($H_f(3+2t)$) →$\mathbb{Z}_3$ defined by

$$c(v)=\begin{cases}2 & w_{2k} \in \ell_2 \text{ for } k = 1,2,3 \ldots\\ 1 & \text{for } w_{3+2k} \in \ell_2 \ for\ k = 0.1,2 \ldots\\ 0 & \text{otherwise}\end{cases}$$

Then $\sigma(v)=\begin{cases}2 & \text{for } x, v_{2+2k} \in \ell_1 \text{for } k = 1,2 \ldots\\ 1 & \text{for } v_{3+2k} \in \ell_1 \text{ where } k = 0,1,2,3 \ldots\\ 0 & \text{elsewhere}\end{cases}$ here $\sigma(x) \neq \sigma(y)\ \forall\ x, y$ of adjacent vertices in H_f(3+2t) ∴ mc(H_f(3+2t))=3 .

Hence the proof.

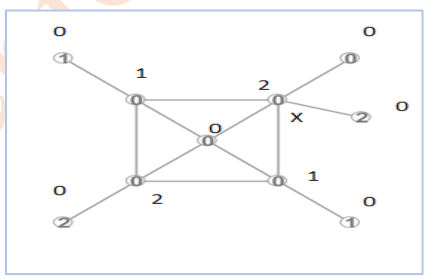

Figure 3.5.1 H_f(5)

(iii) Let v_1 and v_2 fused to become a vertex x in ℓ_1.

Consider a modular coloring c(v): V($H_f(2+2t)$) →$\mathbb{Z}_4$ defined by

$$c(v)=\begin{cases}1 & w_{2k} \in \ell_2 \text{ for } k = 1,2,3 \ldots \\ 2 & \text{for } w_{1+2k} \in \ell_2 \text{ for } k = 0.1,2 \ldots \\ 0 & \text{otherwise}\end{cases}$$

$$\text{Then } \sigma(v) = \begin{cases} 2 & \text{for } v_{3+2k} \in \ell_1 \text{for } k = 0,1,2 \ldots \\ 3 & \text{for } x \in \ell_1 \\ 1 & \text{for } v_{2+2k} \in \ell_1 \text{ where } k = 1,2,3 \ldots \\ 0 & \text{elsewhere} \end{cases}$$

here $\sigma(x) \neq \sigma(y) \; \forall \; x, y$ of adjacent vertices in $H_f(2+2t)$

$\therefore$ mc(H_f(2+2t))=4 . Hence the proof.

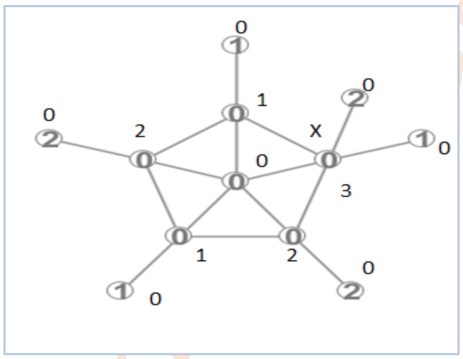

Figure 3.5.2 $H_f(6)$

Fusion of two adjacent vertices one at level ℓ=1 and the other at ℓ=0.

Theorem 3.6:

The graph obtained by fusing two vertices v_1 and u, then

(i) mc(H_f(2))=2.

(ii) mc(H_f(n))=3 for n>2

Proof:

(i) Let v_1 and u fused to become a vertex x.

Consider a modular coloring c(v): V(H_f(2)) →$\mathbb{Z}_2$ defined by c(v)= $\begin{cases} 1 & \text{for } v_2 \in \ell_1 \\ 0 & \text{otherwise} \end{cases}$

Then $\sigma(v) = \begin{cases} 1 & \text{for } w_2 \in \ell_2, x \in \ell_1 \\ 0 & \text{elsewhere} \end{cases}$

here $\sigma(x) \neq \sigma(y)\ \forall\ x, y$ of adjacent vertices in $H_f(2)$

∴ mc(H_f(2))=2. Hence the proof.

(ii)(a) When n is odd, n≥3.

Let v_1 and u fused to become a vertex x.

Consider a modular coloring c(v): V(H_f(n)) →$\mathbb{Z}_3$ defined by

$$c(v)= \begin{cases} 1 & \text{for } x,\ w_{3+2k} \in \ell_2, k = 0,1,2,\dots. \\ 0 & \text{otherwise} \end{cases}$$

Then $\sigma(v) = \begin{cases} 1 & \text{for } w_1 \in \ell_2, v_{2k} \in \ell_1, k = 1,2,3,\dots. \\ 2 & \text{for } v_{3+2k} \in \ell_1, k = 0,1,2,\dots \\ 0 & \text{elsewhere} \end{cases}$

here $\sigma(x) \neq \sigma(y)\ \forall\ x, y$ of adjacent vertices in $H_f(n)$

∴ mc(H_f(n))=3 for n≥3. Hence the proof.

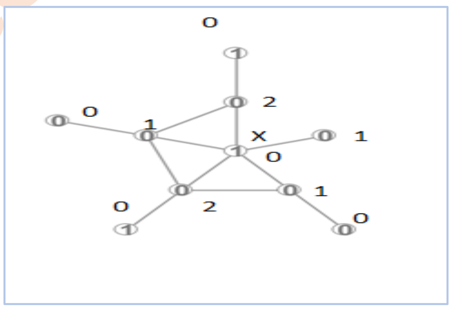

Figure 3.6.1 $H_f(5)$

(ii)(b) When n is even, n≥3.

Let v_1 and u fused to become a vertex x.

Consider a modular coloring c(v): V(H_f(n)) $\rightarrow \mathbb{Z}_3$ defined by

$$c(v)=\begin{cases}1 & \text{for } x,\ w_{2k} \in \ell_2, k = 1,2,\dots. \\ 0 & \text{otherwise}\end{cases}$$

$$\text{Then } \sigma(v) = \begin{cases}1 & \text{for } w_1 \in \ell_2, v_{3+2k} \in \ell_1, k = 0,1,2,3,\dots. \\ 2 & \text{for } v_{2k} \in \ell_1, k = 1,2,\dots \\ 0 & \text{elsewhere}\end{cases}$$

here $\sigma(x) \neq \sigma(y)\ \forall\ x, y$ of adjacent vertices in H_f(n)

∴ mc(H_f(n))=3 for n≥3. Hence the proof.

4. Conclusion

In Helm graph the modular chromatic number of a graph obtained after the fusion of two vertices from different levels, we get mc(H_f(n)) = 2, 3 or 4.But the mc-number of the helm graph is 3 for n≥ 3.In the case of Fan graph mcF_f(n) = mc F(n-1); mcF_f(4k)) =3 and the mc-number of Fan graph is 3. Similarly for a wheel graph mcW_f(W_n)) = mc(W_{n-1}) and the mc-number of wheel graph is 3 or 4 depending on number of vertices is even or odd. The labelling in these different types of graphs shows almost a similarity to one other depending on the level of vertices we are choosing for fusion. Anyway, the chromatic number of graphs before and after the fusion varies from 2 to 4. We cannot expect a higher level of modular chromatic number after fusion of two vertices at different levels. Studying this problem and related problems in the context of fusion of graphs may help in answering the long open question whether all of these problems have a polynomial algorithm. We conclude this paper by listing a number of fusion graph problems of which we do not know the complexity.

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